**Phys/Geol/APS 6610**
**Earth and Planetary Physics I**

**Strings and the Wave Equation**

## 1. Waves on a String: Physical Motivation

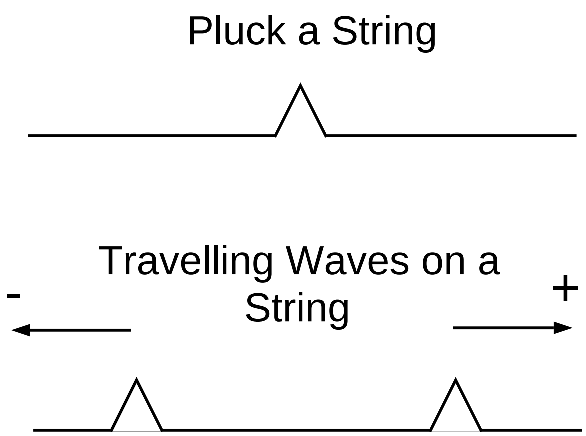


Figure 1: Two traveling waves are set in motion in opposite directions by plucking a string.

Let's begin with a brief review of what you all saw in basic physics as undergrads. Consider waves on a string. When you pluck a string, you send traveling waves in both directions as Figure 1 shows. If the string is finite and clamped at both ends, the traveling waves reflect at the clamping points and superpose to produce standing waves.

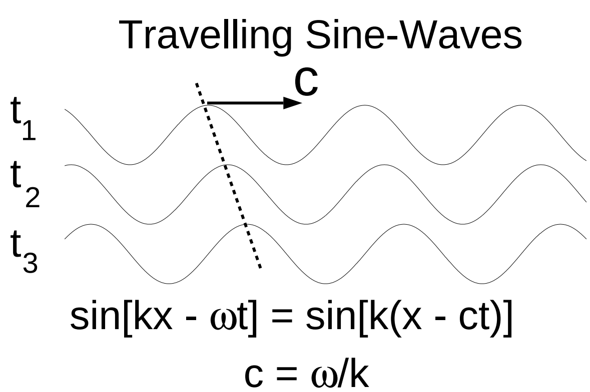


Figure 2: Illustration of a traveling sine-wave. The crest of the wave occurs where the phase is zero, which moves to the right with speed $c = \omega/k$.

Similar to the traveling waves shown in Figure 1, sine-waves can also travel as Figure 2 shows. In fact, if we let $y(x,t)$ denote the displacement of the string in the vertical direction as a function of position along the string ($x$) and time ($t$), it's easy to see that the following expression is a traveling wave

$$y(x,t) = \sin[kx - \omega t] = \sin[k(x - \frac{\omega}{k}t)] = \sin[k(x - ct)] \tag{1}$$

in which each wave-crest or trough moves with speed $c = \omega/k$ to the right. Here $\omega$ is angular frequency ($\omega = 2\pi f$) and $k$ is wavenumber ($k = 2\pi/\lambda$), where $f$ is frequency in Hz and $\lambda$ is wavelength, presumably in meters. To say that a sine-wave moves means that the peaks and troughs move, that is that the locations of constant phase move. The wave given by equation (1), therefore, is seen to move in the $+x$ direction, because the phase $k(x - ct)$ is constant only if $x$ increases as $t$ increases. In fact, it is constant only if $x$ increases with speed $c$. Of course, similar expressions could be derived for a traveling cosine-wave: $\cos[k(x - ct)], \cos[k(x + ct)]$.

We noted above that traveling waves on a clamped string superpose to produce a standing wave. If the traveling waves are sine-waves this is easy to show using a simple trig identity ($\sin(a+b) = \sin a \cos b + \cos a \sin b$). Consider the superposition of two traveling sine-waves and apply the trig identity:

$$\begin{aligned}
\sin(kx - \omega t) \quad + & \quad \sin(kx + \omega t) \\
= & \quad [\sin kx \cos(-\omega t) + \cos kx \sin(-\omega t)] + [\sin kx \cos(\omega t) + \cos kx \sin(\omega t)] \\
= & \quad [\sin kx \cos(\omega t) - \cos kx \sin(\omega t)] + [\sin kx \cos(\omega t) + \cos kx \sin(\omega t)] \\
= & \quad 2 \sin kx \cos \omega t, \qquad (2)
\end{aligned}$$

where we have also used the fact that cosine is a symmetric (or even) function around the origin and sine is anti-symmetric (or odd) so $\cos(-\omega t) = \cos(\omega t)$ and $\sin(-\omega t) = -\sin(\omega t)$. It should be apparent that $\sin kx \sin \omega t$ is also a standing wave, but is shifted $90^\circ$ (half a period) in time.

As equation (1) is the classical form for a traveling sine-wave, equation (2) is the form of a standing sine-wave. Note that there is no standing cosine ($\cos kx \sin \omega t, \cos kx \cos \omega t$) for a clamped string, because the cosine does not satisfy the *boundary conditions* that displacement goes to zero at the ends of the string. If the string would be unclamped at one end, then the standing cosine would be allowed.

From equation (2), we see that standing waves on a string are the product of a spatial shape ($\sin kx$) and a temporal harmonic or oscillation ($\cos \omega t, \sin \omega t$). The shape is sometimes called the *eigenfunction*. To this point, we've merely posited the shape being a sine and ruled out a cosine by consideration of the boundary conditions, but later in the notes we will derive the shape of the string. The shapes of several sines are shown in Figure 3. Each of these potential shapes of oscillation is called a *normal mode* or a *mode of oscillation* and can be denoted by a single number $n$, sometimes called the *quantum number* of the normal mode because the allowed shapes are discrete. The quantum numbers are integers $n = 1, 2, 3, \ldots$ where $n = 1$ is the fundamental mode and $n \geq 2$ are the higher modes of oscillation. Inspection of Figure 3 shows that the wavelength of the fundamental mode is $\lambda_1 = 2a$ and the first higher modes are $\lambda_2 = a, \lambda_3 = 2a/3$, etc., and in general satisfy the following criterion:

$$\lambda_n = \frac{2a}{n}$$

from which the following can be deduced:

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{a} \qquad \omega_n = ck_n = \frac{n\pi c}{a} \tag{3}$$

Because the boundary conditions determine the wavelengths, they also determine the frequencies. This fact is commonly summarized by reporting that the boundary conditions are what determine the frequencies of oscillation, or *eigenfrequencies*.

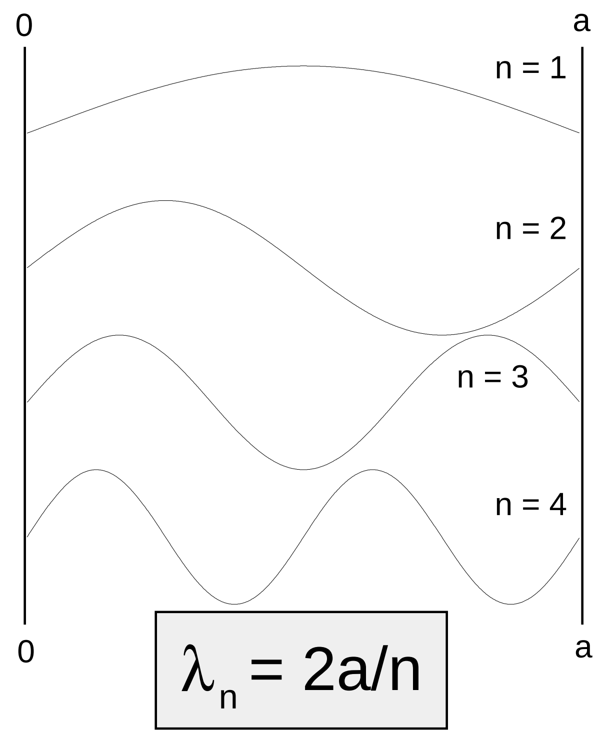


Figure 3: Modes of oscillation of a string clamped at both ends. Each mode has a shape of $\sin(n\pi x/a)$ with wavelength $2a/n$, where $n$ is the modal index or quantum number that specifies the mode.

Note that we have shown that if each standing wave or normal mode on a string, $y_n(x,t)$, is the sum of two traveling waves then it is simply the product of a spatial shape and a temporal oscillation. Let's represent the spatial shape and temporal oscillation as $Y_n(x)$ and $T_n(t)$ so that:

$$y_n(x,t) = Y_n(x)T_n(t) = \sin k_n x \left(A_n \cos\omega_n t + B_n \sin\omega_n t\right). \tag{4}$$

This is actually a rather powerful result, and doesn't hold for all phenomena, and in fact only holds for the string under certain restrictive conditions that we have implicitly assumed here (e.g., the equilibrium tension in the string does not change with time). This result, equation (4), is called the *separation of variables* and we'll use it later in solving the string equation more formally.

The actual displacement that the string would undergo if plucked or kicked would be a sum or superposition of the modes of oscillation as follows:

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} Y_n(x)T_n(t) = \sum_{n=1}^{\infty} \sin(k_n x)\left(A_n \cos\omega_n t + B_n \sin\omega_n t\right) \\ &= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\left(A_n \cos\frac{n\pi ct}{a} + B_n \sin\frac{n\pi ct}{a}\right). \end{aligned} \tag{5}$$

Each coefficient $A_n$ and $B_n$ is a weight that determines both the relative contribution of each mode of oscillation to the final displacement and the phase of the temporal oscillation. These coefficients depend on how the string is set into motion; if it is plucked or kicked, for example.

If, for example, you pluck a string near the node of a mode of oscillation, you will not excite that mode.

It is important to know that the way in which the string is set into motion is called the *initial conditions* and the initial conditions are what determine the $A_n$ and $B_n$. Finding the $A_n$ and $B_n$ is easy if you know about Fourier Series, although it can be rather tedious. The initial shape of the string can be seen from equation (5) to be just the displacement at $t = 0$: $y(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x/a$. This is simply the Fourier Series expansion of the initial displacement pattern of the string. So, if you can find the Fourier Series expansion of the initial displacement pattern, you have the $A_n$. Similarly, you can find the $B_n$ from the initial velocity applied to the string, except you will need to take the Fourier Series expansion of the initial velocity pattern of the string, which is the time derivative of equation (5).

## 2. A Differential Equation You've Seen Before

The wave equation for a string is a *differential equation*. An example that you've seen before is the simple harmonic oscillator (Figure 4).

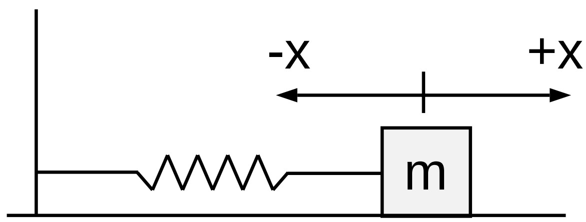


Figure 4: Schematic representation of a simple harmonic oscillator (SHO), in which a mass $m$, connected connected to a spring with spring-constant $\kappa$, oscillates with displacement $\pm x$ about equilibrium.

For small displacements its motion can be modeled with Hooke's Law that says that the force is in the direction opposite to the displacement from equilibrium and has a magnitude proportional to the displacement ($F = -\kappa x$). When this is placed into Newtons' second law ($F = ma$) you get a differential equation as shown here:

$$\begin{aligned} m\frac{d^2x(t)}{dt^2} = ma = F &= -\kappa x(t) \\ \frac{d^2x(t)}{dt^2} + \left(\frac{\kappa}{m}\right)x(t) = \frac{d^2x(t)}{dt^2} + \omega^2 x(t) &= 0 \end{aligned} \tag{6}$$

Equation (6) is sometimes called the simple harmonic oscillator (SHO) equation. The SHO, as you recall, oscillates with frequency $\omega = \sqrt{(\kappa/m)}$. In the parlance of differential equations, it is a *linear*, *second-order*, *homogeneous*, *ordinary differential equation* with *constant coefficients*. It is

- a differential equation because there are derivatives in it,
- ordinary because there are no partial derivatives in it (more on this later),
- second-order because its highest derivative with respect to the independent variable $t$ is of second-order,

- homogeneous because the right-hand-side of the equation is zero which means physically that there are no applied forces, and, finally,
- it has constant coefficients because the terms that multiply the functions of $x$ are constant – in this case it's $\omega^2 = \kappa/m$.

Because this is a linear ODE, if $x_1(t)$ and $x_2(t)$ are solutions, so is $ax_1(t) + bx_2(t)$ where $a, b$ are arbitrary constants. Because it is a second-order ODE, there are two and only two independent solutions. Also within the parlance of differential equations, equations like equation (6) are called *Helmholtz equations.* Ordinary differential equations are often called ODEs.

Helmholtz equations, like the SHO-equation, are particularly easy to solve. The trial solution can be written in a variety of equivalent ways, one of which is:

$$x(t) = A\cos\omega t + B\sin\omega t, \tag{7}$$

where $\omega^2 = \kappa/m$ and $A$ and $B$ are arbitrary constants that depend on the initial conditions – that is on how the oscillator has been set into motion (drag and let go or a kick, for example). Note that there are two independent solutions ($\cos\omega t$, $\sin\omega t$) whose linear combination is also a solution. We can show that equation (7) is a solution to equation (6) by direct substitution:

$$\begin{aligned} \frac{dx(t)}{dt} &= -A\omega\sin\omega t + B\omega\cos\omega t \\ \frac{d^2x(t)}{d^2} &= \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t \end{aligned} \tag{8}$$

Substitution of equations (7) and (8) into (6) establishes the result:

$$\frac{d^2x}{dt^2} + \omega^2 x = \left(-A\omega^2\cos\omega t - B\omega^2\sin\omega\right) + \omega^2\left(A\cos\omega t + B\sin\omega t\right) = 0.$$

The procedure that we followed here is actually similar to how differential equations are solved in practice, you guess a solution and see if it works. The guessed solution is often called the *trial solution* or *ansatz*, which is the fancier German name for it and you can use this to impress your friends who don't know any better. Now that we know the solution to Helmholtz equations like the SHO equation, we have a starting point for trial solutions later on.

## 3. Derivation of the 1-D Wave Equation for a String

Consider the displacement applied to a string of length $L$ in a coordinate system shown in Figure 5. The displacement $y$ is a function of time $t$ and the spatial coordinate $x$: $y(x,t)$. Let $\rho$ be the mass density of the string with units of mass per unit length and assume that $\rho$ is constant. Let $T$ be the tension in the string. Tension is a force and assume that when the string is plucked the tension remains constant throughout the string. This is the same as assuming that the displacement is small for a homogeneous string. Also, assume that the force of gravity is much weaker than tension ($\rho Lg << T$) so that it does not affect the motion of the string appreciably and can, therefore, be ignored.

Inspection of Figure 6 shows that the $x$- and $y-$components of force (i.e., tension) can be written as follows:

$$\text{x-tension} \qquad \sum F_x = T\cos\theta_2 - T\cos\theta_1 \tag{9}$$

$$\text{y-tension} \qquad \sum F_y = T\sin\theta_2 - T\sin\theta_1 \tag{10}$$


Figure 5: Coordinate system for a string of length $L$ that will undergo vertical (or transverse) displacements $y(x,t)$.

We're interested in modeling the vertical motion of the string, $y(x,t)$, so we will explore the use of equation (10).

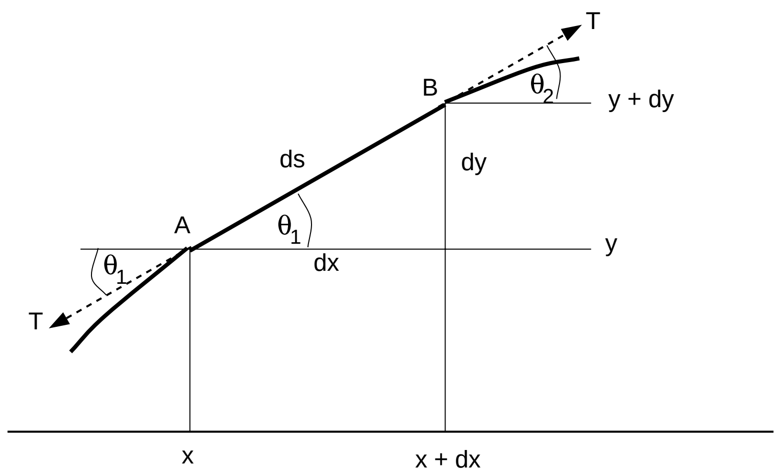


Figure 6: .

Because the oscillations are small, the angles $\theta_1$ and $\theta_2$ are small. Thus, $\sin\theta_1 \approx \tan\theta_1$ and $\sin\theta_2 \approx \tan\theta_2$. We note that the increment of mass in length $ds$ is just $m = \rho ds$. Again, because the oscillations are small, $ds \approx dx$ so $m \approx \rho dx$. We can, therefore, rewrite equation (10) as Newton's second law governing motion of the string in the $y$-direction:

$$ma_y = \rho dx \frac{\partial^2 y}{\partial t^2} = \sum F_y = T\tan\theta_2 - T\tan\theta_1 = T\left(\tan\theta_2 - \tan\theta_1\right). \tag{11}$$

Inspection of Figure 6 again reveals that $\tan\theta \approx \partial y/\partial x$, thus we can rewrite equation (11) as:

$$\rho dx \frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right] \tag{12}$$

Note that the slope of the string at point $B$ can be expressed as a truncated Taylor Series expansion about point $A$:

$$\left(\frac{\partial y}{\partial x}\right)_B \approx \left(\frac{\partial y}{\partial x}\right)_A + \left(\frac{\partial^2 y}{\partial x^2}\right)_A dx. \tag{13}$$

Therefore,

$$\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A = \left(\frac{\partial^2 y}{\partial x^2}\right)_A dx. \tag{14}$$

Substituting equation (14) into equation (12), therefore, reveals that:

$$\rho \frac{\partial^2 y}{\partial t^2} = T\left(\frac{\partial^2 y}{\partial x^2}\right), \tag{15}$$

where we cancelled the factor of $dx$ on both sides of the equation. This equation holds at any location $A$ along the string, so we have removed the subscript $A$ and make location implicit in the function $y(x,t)$.

Both sides of equation (15) have units of force per unit length. If there are forces $F(t)$ applied to the string, they will be added to the right-hand-side of this equation, as follows:

$$\rho \frac{\partial^2 y}{\partial t^2} = T\left(\frac{\partial^2 y}{\partial x^2}\right) + F(t), \tag{16}$$

where $F(t)$ has units of force per unit length.

Remember that the LHS is analogous to $ma$ in Newton's second law so that the RHS represents the forces on the string. The first term on the RHS is the restoring force exerted on the displacement by the string itself. In the absence of applied forces (after the initial conditions), we get the equation for the free oscillations of a homogeneous string which can rewritten as:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \tag{17}$$

where $c = \sqrt{T/\rho}$ is the speed of propagation of the wave traveling in the $\pm x$-direction.

Equation (17) is generally referred to as the 1-D wave equation. It says that the curvature of the string at spatio-temporal point $(x,t)$ is proporational to the vertical acceleration of the string at that point and that the constant of proportionality is related to the horizontal speed of propagation of a wave on the string. I don't know about you, but I wouldn't have guessed that.

All of this holds if the string is homogeneous, that is if the density and tension and, hence, the speed of a wave on the string are constant. If $T$ and $\rho$ are a function of position along the string, then the wave equation is

$$\rho(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right). \tag{18}$$

We will consider different methods to treat this case later on.

Here we have considerd transvese oscillations. For longitudinal oscillations, $u(x)$, the result will be the same but the derivation will differ. For longitudinal waves, relace tension $T(x)$ with Young's modulus $k(x)$, which is analogous to the spring constant for the simple harmonic oscillator. For the longitudinal oscillations of an inhomogeneous string, therefore:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right). \tag{19}$$

In this derivation, stress is assumed proportional to strain (Hooke's Law) and the constant of proportionality is $k$.

## 4. Solving the 1D Homogeneous Wave Equation with Separation of Variables

We now want to solve the wave equation in 1 spatial dimension (1-D), equation (17). This equation governs wave propagation in a 1-D medium, such as a string or a wire.

Partial differential equations such as equation (17) are usually not solved directly, but are transformed into other equations that can be solved. Usually they are transformed first into a set of ODEs, one for each free variable. For the 1-D wave equation, therefore, we'll expect two equations, one in $x$ and one in $t$. The method we're going to follow now is called the *method of separation of variables.*

Equation (17) can be separated into these two constitutive equations by using the method of separation of variables in the following way. Let us assume that the solution can be written (as we know it can for a string) in terms of the product of two functions, one in $x$ and the other in $t$, in the following way:

$$y(x,t) = Y(x)T(t) \tag{20}$$

$Y(x)$ and $T(t)$ are the unknowns we wish to find and equation (20) is a a kind of trial solution and we'll see if it works. To substitute equation (20) into equation (17) we'll first need the space and time derivatives of $y$:

$$\begin{aligned}
\frac{\partial y(x,t)}{\partial x} &= T(t)\frac{\partial Y(x)}{\partial x} = T(t)\frac{dY(x)}{dx} \\
\frac{\partial^2 y(x,t)}{\partial x^2} &= T(t)\frac{\partial^2 Y(x)}{\partial x^2} = T(t)\frac{d^2Y(x)}{dx^2} \qquad (21) \\
\frac{\partial y(x,t)}{\partial t} &= Y(x)\frac{\partial T(t)}{\partial t} = Y(x)\frac{dT(t)}{dt} \\
\frac{\partial^2 y(x,t)}{\partial t^2} &= Y(x)\frac{\partial^2 T(t)}{\partial t^2} = Y(x)\frac{d^2T(t)}{dt^2} \qquad (22)
\end{aligned}$$

Note that we've replaced the partial derivatives on the right-hand side with total derivatives because they are derivatives of functions of a single variable. Substituting equations (21) and (22) into equation (17) we get:

$$\frac{d^2T(t)}{dt^2}Y(x) = c^2\frac{d^2Y(x)}{dx^2}T(t)$$

which upon rearranging yields:

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} \tag{23}$$

Note that the left-hand side of equation (23) is just a function of $t$ and the right-hand side is only a function of $x$.

**Now, comes the key step.** It's simple, but you have to pay attention. How can a function of $t$, which in principle could be changing arbitrarily in time, be equal to a function of $x$ that may be changing arbitrarily in space? Well, to make a long story short, the only way is if both sides of equation (23) are equal to the same constant which is called the *separation constant.* For a reason that will become apparent later, let's let that constant be called $-k^2$, so:

$$\begin{aligned}
\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} &= -k^2 \\
\frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} &= -k^2
\end{aligned}$$

which after a little rearranging can be rewritten as:

$$\frac{d^2Y(x)}{dx^2} + k^2Y(x) = 0 \tag{24}$$

$$\frac{d^2T(t)}{dt^2} + c^2k^2T(t) = 0 \Rightarrow \frac{d^2T(t)}{dt^2} + \omega^2T(t) = 0 \tag{25}$$

where the latter result in equation (25) holds because $\omega = ck$.

Equations (24) and (25) are the two ODEs whose solutions, $Y(x)$ and $T(t)$, can be substituted into equation (20) to give a solution to the PDE, the wave equation given by equation (17). Comparison of equations (24) and (25) with equation (6) reveals that both of these equations are simply Helmholtz equations, which we know how to solve because of their role in the SHO. Their solutions, therefore, are simply:

$$Y(x) = A\cos kx + B\sin kx \tag{26}$$

$$T(t) = C\cos\omega t + D\sin\omega t \tag{27}$$

where $A, B, C$, and $D$ are arbitrary constants. You can see why we defined the separation constant as $-k^2$ because doing so yields equation (26) where $k$ plays the role of wavenumber as we have defined it previously.

The boundary conditions allow us to find $A$ as well as $k$ and, hence, $\omega$ as we will now show. The initial conditions will specify the products $BC$ and $BD$. This is discussed further in the next section.

Now, let's apply the boundary conditions. Assume that the string is clamped both at both ends: $x = 0$ and $x = a$. The boundary conditions, therefore, are $y(0,t) = y(a,t) = 0$ or equivalently $Y(0) = Y(a) = 0$, so using equations (26) and (27) we see that:

$$0 = Y(0) = A\cos(0) + B\sin(0) \Rightarrow A = 0 \tag{28}$$

$$0 = Y(a) = B\sin ka \Rightarrow k = \frac{1}{a}\sin^{-1}(0) \Rightarrow k_n = \frac{n\pi}{a}, \tag{29}$$

where $n$ is an integer. Remember that the expression $\sin^{-1}(0)$ should be read as the angle(s) at which sine is zero; which is just multiples of $\pi$.

We see, therefore, that we've established that there are a countably infinite number of allowable separation constants $k$ indexed by the number $n$, that we recognize as the mode number or quantum number as discussed above. In section 1, we established that $k_n = n\pi/a$ based on purely physical considerations, here the reasoning was more mathematical but the result is the same. We see now that:

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{a} \qquad \omega_n = ck_n = \frac{n\pi c}{a}, \tag{30}$$

which is the same as equations (10) above. You can see through equations (28) and (29) how the boundary conditions determine the frequencies of oscillation in practice.

The final solution $y(x,t)$ is a linear combination of all of the solutions indexed by $n$:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} Y_n(x)T_n(t) = \sum_{n=1}^{\infty} B_n \sin k_n x\,(C_n\cos\omega_n t + D_n \sin\omega_n t) \tag{31}$$

$$= \sum_{n=1}^{\infty} \sin k_n x\,(A'_n\cos\omega_n t + B'_n\sin\omega_n t) = \sum_{n=1}^{\infty} C'_n \sin k_n x\,(\sin(\omega_n t - \phi_n)) \tag{32}$$

where we recombined the three arbitrary constants into two ($A'_n \equiv B_n C_n$ and $B'_n \equiv B_n D_n$) and also rewritten in terms of a phase shift $\phi_n$ which we will reference in the discussion of energy below. This reproduces the physically motivated equation (5) above. As before, the initial conditions will determine the coefficients ($A'_n$, $B'_n$) or ($C'_n$, $\phi_n$).

**5. Application of Initial Conditions**

For a string clamped at both ends, the the solution for displacement $y(x,t)$, dropping the primes on the coefficients, is:

$$y(x,t) = \sum_{n=1}^{\infty} \sin k_n x \left(A_n \cos \omega t + B_n \sin \omega t\right), \tag{33}$$

where the coefficients $A_n$ and $B_n$ depend on how the string is set into motion, i.e., on the initial conditions, and $k_n = n\pi/L$ and $\omega_n = ck_n$ where $L$ is the length of the string and $c$ is the speed of propagation of waves on the string.

If $f(x)$ and $g(x)$ are the initial patterns of displacement and velocity imparted to the string, then from equation (33) we see that:

$$y(x,0) \;=\; f(x) = \sum_{n=1}^{\infty} A_n \sin k_n x = \sum_{n=1}^{\infty} a_n \sin k_n x, \tag{34}$$

$$v(x,0) = \dot{y}(x,0) \;=\; g(x) = \sum_{n=1}^{\infty} \omega_n B_n \sin k_n x = \sum_{n=1}^{\infty} b_n \sin k_n x. \tag{35}$$

The final equality in equations (34) and (35) is just the expansion of $f(x)$ and $g(x)$ in a Fourier Series. In both cases, the Fourier Series is only a sine-series because the boundary conditions require that the function go to zero at the end-points ($x = 0, x = L$). As usual, the coefficients in the Fourier Series are given by:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)dx, \tag{36}$$

$$b_n = \frac{2}{L}\int_0^L g(x)\sin(k_n x)dx, \tag{37}$$

Here the constant in front of the integral is $2/L$ rather than $1/L$ because of interval we're considering goes from 0 to $L$ rather than $-L/2$ to $L/2$. Comparison of equations (34) and (35) with (36) and (37) reveals that:

$$A_n \;=\; a_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)dx, \tag{38}$$

$$B_n \;=\; \frac{b_n}{\omega_n} = \frac{2}{\omega_n L}\int_0^L g(x)\sin(k_n x)dx. \tag{39}$$

These equations together with equation (33) give the solution to the problem with the initial conditions imposed.

**Example:** Let $y(x,0) = f(x) = y_0 \sin(2\pi x/L)$ and $\dot{y}(x,0) = g(x) = 0$. Then $a_n = y_0 \delta_{n2}$ and $b_n = 0$ so

$$y(x,t) = y_0 \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi ct}{L}\right). \tag{40}$$

## 6. D'Alembert's Solution to the 1-D Wave Equation

To this point, we expressed the solution to the 1-D wave equation as a sum of standing waves. This is called the solution in terms of *normal modes* or *Fourier* basis functions. There is another approach in terms of *traveling waves* that is attributable originally to *d'Alembert*.

The idea is to try to find a pair of independent variables that transforms the 1-D wave equation, equation (17), into a simpler equation. Let's try a linear transformation:

$$\xi = x + at \qquad \eta = x + bt, \tag{41}$$

where $a$ and $b$ are to be determined. We want to rephrase the string equation in $y(x,t)$ in terms of $y(\xi,\eta)$. To do this we need to find $\partial^2 y/\partial x^2$ and $\partial^2 y/\partial t^2$ in terms of derivatives in $\xi$ and $\eta$:

$$\begin{aligned}
\frac{\partial y(\xi,\eta)}{\partial x} &= \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial y}{\partial \xi} + \frac{\partial \eta}{\partial x} && (42)\\
\frac{\partial y^2(\xi,\eta)}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial \xi} + \frac{\partial \eta}{\partial x}\right) && (43)\\
&= \frac{\partial^2 y}{\partial \xi^2}\frac{\partial \eta}{\partial x} + \frac{\partial^2 y}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial^2 y}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 y}{\partial \eta^2}\frac{\partial \eta}{\partial x} && (44)\\
&= \frac{\partial^2 y}{\partial \xi^2} + 2\frac{\partial^2 y}{\partial \xi \partial \eta} + \frac{\partial^2 y}{\partial \eta^2} && (45)
\end{aligned}$$

Through a similar derivation we can show that

$$\begin{aligned}
\frac{\partial y(\xi,\eta)}{\partial t} &= a\frac{\partial y}{\partial \xi} + b\frac{\partial y}{\partial \eta} && (46)\\
\frac{\partial^2 y(\xi,\eta)}{\partial t^2} &= a^2\frac{\partial^2 y}{\partial \xi^2} + 2ab\frac{\partial^2 y}{\partial \xi \partial \eta} + b^2\frac{\partial^2 y}{\partial \eta^2}. && (47)
\end{aligned}$$

Substituting equations (45) and (47) into equation the string equation, equation (17), we get:

$$\left(1 - \frac{a^2}{c^2}\right)\frac{\partial^2 y}{\partial \xi^2} + 2\left(1 - \frac{ab}{c^2}\right)\frac{\partial^2 y}{\partial \xi \partial \eta} + \left(1 - \frac{b^2}{c^2}\right)\frac{\partial \cdot y}{\partial \eta^2} = 0. \tag{48}$$

We can choose $a$ and $b$ to be anything we want, as long as it makes the resulting equation easy to solve. Let's set $a^2 = b^2 = c^2$ so that the first and third terms in equation (48) disappear and also choose $a$ ane $b$ to have opposite signs, to ensure that $\partial^2 y/\partial \xi \partial \eta \neq 0$. The result is:

$$\frac{\partial^2 y}{\partial \xi \partial \eta} = 0. \tag{49}$$

where

$$\xi = x - ct \qquad \eta = x + ct. \tag{50}$$

Equations (49) and (50) are equivalent to equation (17). Equation (50) is a linear transformation that takes equation (17) into equation (49).

To solve equation (49) we first integrate with respect to $\eta$, which gives us an arbitrary "constant" which is actually an arbitrary function of $\xi$:

$$\frac{\partial y}{\partial \xi} = h(\xi). \tag{51}$$

We now integrate again with respect to $\xi$ where the arbitary "constant" of integration is a function of $\eta$. The result then is:

$$y(\xi,\eta) = f(\xi) + g(\eta) \tag{52}$$

$$y(x,t) = f(x-ct) + g(x+ct). \tag{53}$$

Equation (53) is the result we wanted. It says that the general solution to the 1-D wave equation is a function traveling to the right at speed $c$, $f(x-ct))$, and another one traveling to the left at speed $c$, $g(x+ct)$. The entire displacement of the string is a superposition of these two waves.

The functions $f$ and $g$ can be found by specifying the initial displacement and velocity of the string. For example, suppose that the initial displacement is some function $\phi(x)$ and its initial velocity is $\psi(x)$ so that

$$y(x,0) = \phi(x) \qquad\qquad \dot{y}(x,0) = \psi(x). \tag{54}$$

Then,

$$\phi(x) = f(x) + g(x) \tag{55}$$

$$\psi(x) = -cf'(x) + cg'(x), \tag{56}$$

where the primes mean differentiation with respect to $x$. Integrating equation (56) from 0 to $x$:

$$f(x) - g(x) = -\frac{1}{c}\int_0^x \psi(x')dx' + a, \tag{57}$$

where the constant of integration $a = f(0) - g(0)$. Adding and subtracting this to equation (55) gives:

$$f(x) = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(x')dx' + \frac{a}{2}, \tag{58}$$

$$g(x) = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(x')dx' - \frac{a}{2}. \tag{59}$$

Therefore,

$$y(x,t) = f(x-ct) + g(x+ct) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \int_{x-ct}^{x+ct} \psi(x')dx'. \tag{60}$$

Equation (60) is d'Alembert's solution to the 1-D wave equation. It gives the displacement $y(x,t)$ in terms of the given initial conditions on displacement, $y(x,0) = \phi(x)$, and velocity, $\dot{y}(x,0) = \psi(x)$.

**Example 1:** Suppose that $\psi(x) = u(x,0) = 1 - |x|$ for $-1 \le x \le 1$ and is 0 otherwise, and that $\psi(x) = \dot{y}(x,0) = 0$. That is, assume the string starts from rest but is deformed around the origin by a triangular diplacement pattern. As time increases, the displacement will be given by:

$$y(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right]. \tag{61}$$

The original triangular waveform splits into two triangles each with half the amplitude of the original one, with half moving to the right with speed $+c$ and half moving to the left with speed $-c$. So, if the string is simply displaced from equilibrium, the waves that propagate on the string will look like the original displacement. If the string has a non-zero original velocity, however, the propagating displacement pattern will differ from the original displacement.

**Example 2:** As a second example, consider the following:

$$y(x,t) = Ce^{i(\omega t - kx)} + C^* e^{-i(\omega t - kx)}, \tag{62}$$

where $*$ denotes complex conjugation, which is necessary to ensure a real function. This is called a plane wave solution, and is useful since an arbitrary function with only a finite number of discontinuities can be expressed as a sum of such plane waves. This is just a 2D Fourier Series.

Substituting equation (62) into (17) shows that the solution is acceptable provided that $\omega$ and $k$ satisfy $\omega = \pm ck$ – the dispersion relation again. For sinusoidal traveling waves such as in equation (62), displacement patterns repeat. Points at which the displacement amplitudes are equal have equal phases; i.e. $\omega t - kx = \omega(t + \Delta t) - k(x + \Delta x)$. This is true if $\omega \Delta t - k \Delta x = 0 \rightarrow c = dx/dt = \omega/k$. This is the velocity of a phase, so is called the phase velocity. Wave groups are more complicated and we'll get back to them later, but they are constructed out of a set of component waves such as plane waves and satisfy the condition of constructive interference; that is each component must have the same value of the phase angle $\omega t - kx + \phi$ although the individual values of $\omega, k$ and $\phi$ may be different. Thus, the quantity $\omega t - kx + \phi$ must be independent of frequency if evaluated at a characteristic frequency, $\omega_0$, of the group: $d(\omega t - kx + \phi)/d\omega|_{\omega_0} = 0$. Carrying out the differential we find that the constructive interference condition will be met for a wave traveling with the group velocity $U = d\omega/dk|_{\omega_0}$. Thus, the group velocity is just the slope of the dispersion relation. For a nondispersive system, the dispersion relation is linear and the group velocity is constant with frequency. For the 1D homogeneous string, the slope of the dispersion relation is just $c$, and therefore $U = c$. It is not true for all nondispersive systems that group and phase velocities are equal.

As an historical aside, D'Alembert hoped that this method of solution to the 1-D wave equation would be applicable to other PDEs, but that's not the case.

The point of this exercise has been to show that there are two types of basis functions that are useful in studying waves on a string: the Fourier type (standing waves, sines and cosines) and the d'Alembert type (traveling waves). In a string, we see (and hear) the modes of oscillation and the waves themselves are obscure. However, in the Earth we see individual packets of energy arriving on a seismogram. Although we will frequently use modal techniques to solve difficult problems, we frequently see and identify waves in data and, therefore, think of modes superposing to produce the waves. A facile geophysicist must be able to think in both modes and waves as he or she must be able to think both in the time and frequency domains.

## 7. Energy and its Conservation

*Energy*

Expressions for the kinetic and potential energy density are required for Lagrangian and Hamiltonian dynamics, and we will consider them briefly here. The energy density of a string is its energy per unit length. Let $\mathcal{K}$ denote kinetic energy density, then:

$$\mathcal{K} = \frac{1}{2}\rho v^2 = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2, \tag{63}$$

and the total kinetic energy, $K$, is:

$$K = \frac{1}{2}\int_0^L \rho\left(\frac{\partial y}{\partial t}\right)^2 dx. \tag{64}$$

Note that here we use $K$ rather than $T$ for kinetic energy so as not to confuse it with tension. Italicized capital letters distinguish quantities that have units of energy density from those having units of energy.

The derivation of the appropriate expression for potential energy is a little more complicated. Consider an increment of the string, $dx$, stretched to a new length $ds$. The change in length is just $ds - dx$. This derivation is a little easier if we consider transverse rather than longitudinal oscillations. The final equations are the same if tension, $T$, is replaced with Young's modulus, $k$. So imagine a transverse perturbation, $dy$. Then, $ds^2 = dx^2 + dy^2$ and factoring out a $dx$:

$$ds - dx = dx\left[\sqrt{1 + (dy/dx)^2} - 1\right]. \tag{65}$$

Under the small oscillation approximation, $dy/dx << 1$, so we can Taylor expand the square root in equation (65) which yields:

$$ds - dx = \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx. \tag{66}$$

Stretching takes place against tension, and the work against tension is just tension times stretch:

$$W = \frac{1}{2}T\left(\frac{dy}{dx}\right)^2 dx, \tag{67}$$

and the potential energy density, $\mathcal{V}$, is:

$$\mathcal{V} = \frac{1}{2}T\left(\frac{dy}{dx}\right)^2, \tag{68}$$

so that the total potential energy becomes:

$$V = \frac{1}{2}\int_0^L T\left(\frac{dy}{dx}\right)^2 dx. \tag{69}$$

Total energy density $\mathcal{H} = \mathcal{K} + \mathcal{V}$, so that

$$\mathcal{H} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{dy}{dx}\right)^2. \tag{70}$$

and the total energy can be written

$$H = \int_0^L \mathcal{H} dx \tag{71}$$

$$= \frac{1}{2}\rho \int_0^L \left[ \left(\frac{\partial y}{\partial t}\right)^2 + c^2 \left(\frac{dy}{dx}\right)^2 \right] dx. \tag{72}$$

It should be noted that this equation only holds either when the endpoints of the string are fixed or when the spatial gradient of displacement at the endpoints goes to zero. The former is the the situation we are considering here so equation (72) is fine, but it should not be considered a general formula which is applicable in any situation. (For further discussion see Morse and Feshbach ch. 2.1.)

It will be left as an exercise to calculate the energy per mode of oscillation. The kinetic, potential, and total energies in terms of the modes of oscillation from equation (32) is simply:

$$K = \frac{1}{4}\rho L \sum_n \left(\frac{n\pi c}{L}\right)^2 c_n^2 \cos^2 \left(\frac{n\pi ct}{L} - \phi_n\right), \tag{73}$$

$$V = \frac{1}{4}\rho L \sum_n \left(\frac{n\pi c}{L}\right)^2 c_n^2 \sin^2 \left(\frac{n\pi ct}{L} - \phi_n\right), \tag{74}$$

$$H = \frac{1}{4}\rho L \sum_n \left(\frac{n\pi c}{L}\right)^2 c_n^2. \tag{75}$$

Thus, the kinetic and potential energies are out of phase by 90 degrees and sum to give a constant of motion. In each mode, the energy oscillates between kinetic and potential forms as the string itself oscillates. The periods of the oscillations of the string are $2L/nc$, while those of energy are half that great, corresponding to successive realizations of a given phase of the motion.

Equations (73) - (75) demonstrate a striking result, that the individual terms in the Fourier series solution, equation (32), are independent in that each carries a fixed amount of energy and this energy cannot be exchanged with the energies of any other mode, much as the traveling waves in the string do not exchange energy when they pass one another but simply superpose. We say these modes are, therefore, uncoupled.

Complexities added to the string can produce modal coupling as we will see as the course progresses. Later on we will also consider reflection of string waves off fixed masses, the solution to the inhomogeneous string problem, and perturbative and approximate methods for finding frequencies and shapes of oscillations.

*Conservation of Energy*

Although the total energy, $H$, is stationary for the entire string, energy can flow along the string. Thus, an energy flux, $J$, should be defined. The energy flux, $\mathcal{J}$, and the energy density are related by the conservation of energy which states that the time rate of change of energy density at a point is related to the net amount of energy flowing into or out of a region. For example, if more energy flows across the point $x + dx$ than flows across $x$, then the energy contained in the length $dx$ of the string must diminish:

$$\mathcal{J}(x + dx) - \mathcal{J}(x) = -dx \frac{\partial \mathcal{H}}{\partial t}, \tag{76}$$

which upon rewriting becomes

$$\frac{\partial \mathcal{J}}{\partial x} + \frac{\partial \mathcal{H}}{\partial t} = 0. \tag{77}$$

Therefore, a closed form expression for energy flux can be found as follows:

$$\begin{aligned}
\mathcal{J} &= -\int \frac{\partial \mathcal{H}}{\partial t} dx && (78)\\
&= -\frac{1}{2}\rho \int \frac{\partial}{\partial t}\left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{dy}{dx}\right)^2\right] dx && (79)\\
&= -\rho \int \left[\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + \frac{T}{\rho}\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\right] dx && (80)\\
&= -T \int \left[\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\right] dx && (81)\\
&= -T \int \frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\right] dx && (82)\\
&= -T \frac{\partial y}{\partial t}\frac{\partial y}{\partial x}. && (83)
\end{aligned}$$

*Derivation of the String Equation from Conservation of Energy*

Using this expression for energy flux and the expression for total energy density given by equation (70), from the conservation of energy an alternative derivation of the wave equation for the string results:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{J}}{\partial x} + \frac{\partial \mathcal{H}}{\partial t} && (84)\\
&= -T\frac{\partial^2 y}{\partial x \partial t}\frac{\partial y}{\partial x} - T\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial x^2} + \rho\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + T\frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t \partial x} && (85)\\
&= \frac{\partial y}{\partial t}\left(-T\frac{\partial^2 y}{\partial x^2} + \rho\frac{\partial^2 y}{\partial t^2}\right). && (86)
\end{aligned}$$

Since $\partial y/\partial t$ is an arbitrary function, equation (86) implies (17) where $c^2 = T/\rho$. Recall the previous derivation of the string equation came from the application of Newton's Second Law.

## 8. Lagrange's Equation for a String

Fetter and Walecka use Hamilton's Principle to derive Lagrange's equation for a continuous system on pages 128 - 130. This is their equation (25.59), which is as follows:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial(\partial y/\partial x)}\right) = \frac{\partial \mathcal{L}}{\partial y} \tag{87}$$

The second term on the left-hand side is new, it does not appear in Lagrange's equation for discrete systems. Thus, for a continuous system the functional dependence of the Lagrangian is $\mathcal{L}(y, \dot{y}, \partial y/\partial x)$, at least in principle. From our discussion in section 7, we can write down $\mathcal{L}$ explicitly. Instead of using kinetic and potential energies to derive the Lagrangian, let's obtain the Lagrangian density (for simplicity still called $\mathcal{L}$ here) from the kinetic and potential energy densities:

$$\mathcal{L} = \mathcal{K} - \mathcal{V} = \frac{1}{2}\rho(x)\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T(x)\left(\frac{\partial y}{\partial x}\right)^2 \tag{88}$$

Note that by retaining the functional dependence of tension and density on position $x$ along the string, we have the Lagrangian for an inhomogeneous string. You can see immediately that $\mathcal{L}(\dot{y}, \partial y/\partial x)$; that is, that the Lagrangian density is independent of the displacement $y$. In the parlance of Lagrangian mechanics, $y$ is a *cyclic* coordinate. Remember that independence on the amplitude of displacement results from the small amplitude approximation which is related to Hooke's Law. The existence of a cyclic coordinate is a *symmetry principle*, which implies the existence of a conserved quantity which in this case is mechanical energy.

We want to subsitute the Lagrangian for the inhomogeneous string, equation (88), into Lagrange's equation for a continuous system, equation (87). First, take the appropriate derivatives:

$$\begin{aligned}
\partial\mathcal{L}/\partial y &= 0 && (89)\\
\partial\mathcal{L}/\partial\dot{y} &= \rho(x)\dot{y} && (90)\\
\frac{d}{dt}\left(\partial\mathcal{L}/\partial\dot{y}\right) &= \rho(x)\ddot{y} && (91)\\
\partial\mathcal{L}/\partial(\partial y/\partial x) &= T(x)\left(\partial y/\partial x\right) && (92)
\end{aligned}$$

Now, substitute the derivatives into equation (87):

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial(\partial y/\partial x)}\right) \Rightarrow \rho(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\frac{\partial y}{\partial x}\right). \tag{93}$$

Which is the same as the inhomogeneous wave equation we wrote down but did not derive earlier (equation (18).