Physics 2140
Methods in Theoretical Physics
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Application of Initial Conditions in the Solution of the 1-D Wave Equation for a String

Recall the string equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}, \tag{1}$$

where $c = \sqrt{T/\rho}$ is the speed of propagation of the wave travelling in the $\pm x$-direction. For a string clamped at both ends, the solution for displacement $y(x,t)$ is

$$y(x,t) = \sum_{n=1}^{\infty} \sin k_n x \left(A_n \cos \omega t + B_n \sin \omega t\right), \tag{2}$$

where the coefficients A_n and B_n depend on how the string is set into motion, i.e., on the initial conditions, and $k_n = n\pi/L$ and $\omega_n = ck_n$ where L is the length of the string and c is the speed of propagation of waves on the string.

If $f(x)$ and $g(x)$ be the initial patterns of displacement and velocity imparted to the string, then from equation (2) we see that:

$$\begin{aligned} y(x,0) &= f(x) = \sum_{n=1}^{\infty} A_n \sin k_n x = \sum_{n=1}^{\infty} a_n \sin k_n x, & (3)\\ v(x,0) = \dot{y}(x,0) &= g(x) = \sum_{n=1}^{\infty} \omega_n B_n \sin k_n x = \sum_{n=1}^{\infty} b_n \sin k_n x. & (4)\end{aligned}$$

The final equality in equations (3) and (4) is just the expansion of $f(x)$ and $g(x)$ in a Fourier Series. In both cases, the Fourier Series is only a sine-series because the boundary conditions require that the function go to zero at the end-points ($x = 0, x = L$). As usual, the coefficients in the Fourier Series are given by:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)dx, \tag{5}$$

$$b_n = \frac{2}{L}\int_0^L g(x)\sin(k_n x)dx, \tag{6}$$

Here the constant in front of the integral is $2/L$ rather than $1/L$ because of interval we're considering goes from 0 to L rather than $-L/2$ to $L/2$. Comparison of equations (3) and (4) with (5) and (6) reveals that:

$$\begin{aligned} A_n &= a_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)dx, & (7)\\ B_n &= \frac{b_n}{\omega_n} = \frac{2}{\omega_n L}\int_0^L g(x)\sin(k_n x)dx. & (8)\end{aligned}$$

These equations together with equation (2) give the solution to the problem with the initial conditions imposed.

Example: Let $y(x,0) = f(x) = y_0 \sin(2\pi x/L)$ and $\dot{y}(x,0) = g(x) = 0$. Then $a_n = y_0\delta_{n2}$ and $b_n = 0$ so

$$y(x,t) = y_0 \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi ct}{L}\right). \tag{9}$$