Physics 2140
Methods in Theoretical Physics
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Strings and the Wave Equation

In freshman physics, you learned a number of facts about 1-D waves that were physically motivated and presented; for example, waves on strings. The purpose of these notes is to take a deeper look at these results by inspecting the mathematics that gives rise to them. Doing so requires introducing linear partial differential equations and a general method of solving such equations known as the method of separation of variables. Boas talks about this in Chapter 13.

1. Waves on a String: Physical Motivation

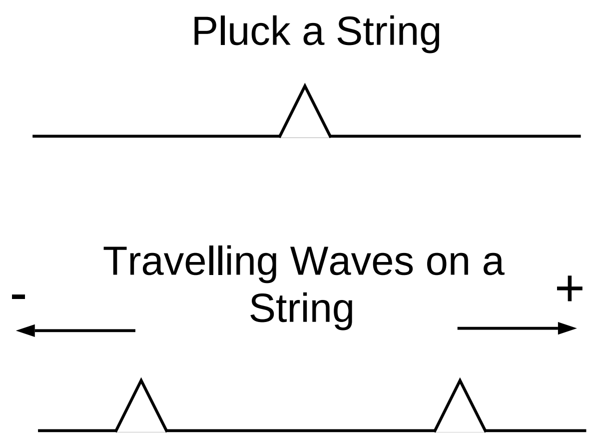

Figure 1: Two travelling waves are set in motion in opposite directions by plucking a string.

Let's begin with a brief review of what you saw in first year physics. Consider waves on a string. When you pluck a string, you send traveling waves in both directions as Figure 1 shows. If the string is finite and clamped at both ends, the traveling waves reflect at the clamping points and superpose to produce standing waves.

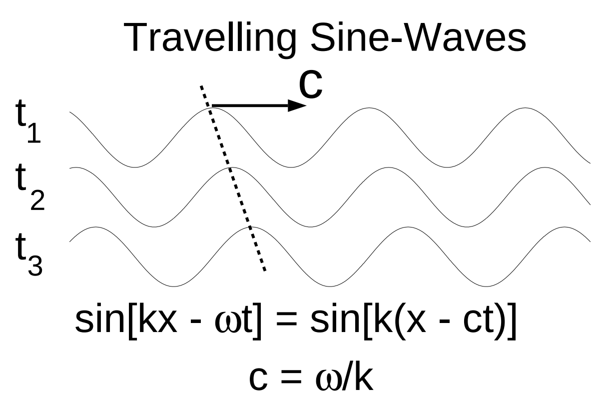

Figure 2: Illustration of a travelling sine-wave. The crest of the wave occurs where the phase is zero, which moves to the right with speed $c = \omega/k$.

Similar to the traveling waves shown in Figure 1, sine-waves can also travel as Figure 2 shows. In fact, if we let $y(x,t)$ denote the displacement of the string in the vertical direction as a function of position along the string (x) and time (t), it's easy to see that the following expression is a traveling wave

$$y(x,t) = \sin[kx - \omega t] = \sin[k(x - \frac{\omega}{k}t)] = \sin[k(x - ct)] \tag{1}$$

in which each wave-crest or trough moves with speed $c = \omega/k$ to the right. Here ω is angular frequency ($\omega = 2\pi f$) and k is wavenumber ($k = 2\pi/\lambda$), where f is frequency in Hz and λ is wavelength, presumably in meters. To say that a sine-wave moves means that the peaks and troughs move, that is that the locations of constant phase move. The wave given by equation (1), therefore, is seen to move in the $+x$ direction, because the phase $k(x-ct)$ is constant only if x increases as t increases. In fact, it is constant only if x increases with speed c. Of course, similar expressions could be derived for a traveling cosine-wave: $\cos[k(x-ct)], \cos[k(x+ct)]$.

We noted above that traveling waves on a clamped string superpose to produce a standing wave. If the traveling waves are sine-waves this is easy to show using a simple trig identity ($\sin(a+b) = \sin a \cos b + \cos a \sin b$). Consider the superposition of two traveling sine-waves and apply the trig identity:

$$\begin{aligned}
\sin(kx - \omega t) \quad + \quad & \sin(kx + \omega t) \\
= \quad & [\sin kx \cos(-\omega t) + \cos kx \sin(-\omega t)] + [\sin kx \cos(\omega t) + \cos kx \sin(\omega t)] \\
= \quad & [\sin kx \cos(\omega t) - \cos kx \sin(\omega t)] + [\sin kx \cos(\omega t) + \cos kx \sin(\omega t)] \\
= \quad & 2 \sin kx \cos \omega t,
\end{aligned} \tag{2}$$

where we have also used the fact that cosine is a symmetric (or even) function around the origin and sine is anti-symmetric (or odd) so $\cos(-\omega t) = \cos(\omega t)$ and $\sin(-\omega t) = -\sin(\omega t)$. It should be apparent that $\sin kx \sin \omega t$ is also a standing wave, but is shifted $90°$ (half a period) in time.

As equation (1) is the classical form for a traveling sine-wave, equation (2) is the form of a standing sine-wave. Note that there is no standing cosine ($\cos kx \sin \omega t, \cos kx \cos \omega t$) for a clamped string, because the cosine does not satisfy the *boundary conditions* that displacement goes to zero at the ends of the string. If the string would be unclamped at one end, then the standing cosine would be allowed.

From equation (2), we see that standing waves on a string are the product of a spatial shape ($\sin kx$) and a temporal harmonic or oscillation ($\cos \omega t, \sin \omega t$). The shape is sometimes called the *eigenfunction*. To this point, we've merely posited the shape being a sine and ruled out a cosine by consideration of the boundary conditions, but later in the notes we will derive the shape of the string. The shapes of several sines are shown in Figure 3. Each of these potential shapes of oscillation is called a *normal mode* or a *mode of oscillation* and can be denoted by a single number n, sometimes called the *quantum number* of the normal mode because the allowed shapes are discrete. The quantum numbers are integers $n = 1, 2, 3, \ldots$ where $n = 1$ is the fundamental mode and $n \geq 2$ are the higher modes of oscillation. Inspection of Figure 3 shows that the wavelength of the fundamental mode is $\lambda_1 = 2a$ and the first higher modes are $\lambda_2 = a, \lambda_3 = 2a/3$, etc., and in general satisfy the following criterion:

$$\lambda_n = \frac{2a}{n}$$

from which the following can be deduced:

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{a} \qquad \omega_n = ck_n = \frac{n\pi c}{a} \tag{3}$$

Because the boundary conditions determine the wavelengths, they also determine the frequencies. This fact is commonly summarized by reporting that the boundary conditions are what determine the frequencies of oscillation, or *eigenfrequencies*.

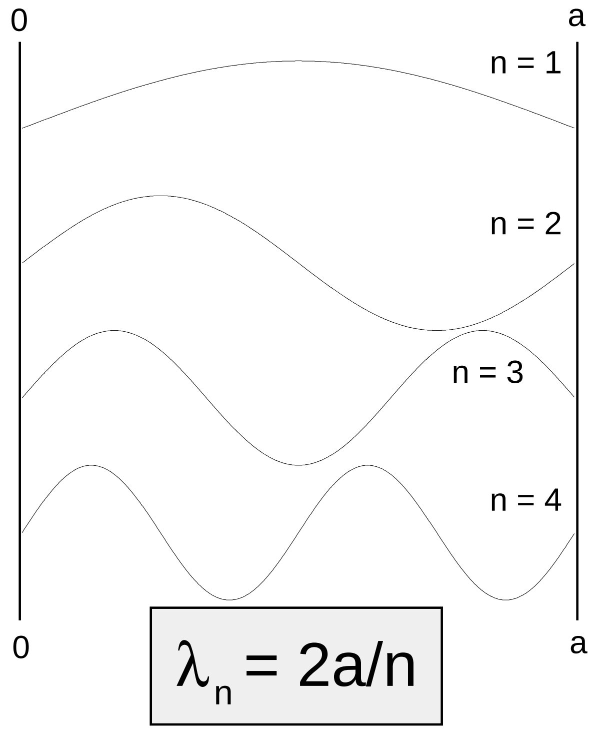

Figure 3: Modes of oscillation of a string clamped at both ends. Each mode has a shape of $\sin(n\pi x/a)$ with wavelength $2a/n$, where n is the modal index or quantum number that specifies the mode.

Note that we have shown that if each standing wave or normal mode on a string, $y_n(x,t)$, is the sum of two traveling waves then it is simply the product of a spatial shape and a temporal oscillation. Let's represent the spatial shape and temporal oscillation as $Y_n(x)$ and $T_n(t)$ so that:

$$y_n(x,t) = Y_n(x)T_n(t) = \sin k_n x\,(A_n \cos\omega_n t + B_n \sin\omega_n t)\,. \tag{4}$$

This is actually a rather powerful result, and doesn't hold for all phenomena, and in fact only holds for the string under certain restrictive conditions that we have implicitly assumed here (e.g., the equilibrium tension in the string does not change with time). This result, equation (4), is called the *separation of variables* and we'll use it later in solving the string equation more formally.

The actual displacement that the string would undergo if plucked or kicked would be a sum or superposition of the modes of oscillation as follows:

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} Y_n(x)T_n(t) = \sum_{n=1}^{\infty} \sin(k_n x)\,(A_n\cos\omega_n t + B_n \sin\omega_n t) \\ &= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{a}\left(A_n \cos\frac{n\pi ct}{a} + B_n \sin\frac{n\pi ct}{a}\right). \end{aligned} \tag{5}$$

Each coefficient A_n and B_n is a weight that determines both the relative contribution of each mode of oscillation to the final displacement and the phase of the temporal oscillation. These coefficients depend on how the string is set into motion; if it is plucked or kicked, for example. If, for example, you pluck a string near the node of a mode of oscillation, you will not excite that mode.

It is important to know that the way in which the string is set into motion is called the *initial conditions* and the initial conditions are what determine the A_n and B_n. Finding the

A_n and B_n is easy if you know about Fourier Series, although it can be rather tedious. The initial shape of the string can be seen from equation (5) to be just the displacement at $t = 0$: $y(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x/a$. This is simply the Fourier Series expansion of the initial displacement pattern of the string. So, if you can find the Fourier Series expansion of the initial displacement pattern, you have the A_n. Similarly, you can find the B_n from the initial velocity applied to the string, except you will need to take the Fourier Series expansion of the initial velocity pattern of the string, which is the time derivative of equation (5).

2. A Differential Equation You've Seen Before

The wave equation for a string is a *differential equation.* An example that you've seen before is the simple harmonic oscillator (Figure 4).

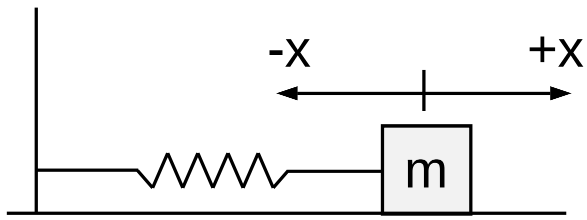

Figure 4: Schematic representation of a simple harmonic oscillator (SHO), in which a mass m, connected connected to a spring with spring-constant κ, oscillates with displacement $\pm x$ about equilibrium.

For small displacements its motion can be modeled with Hooke's Law that says that the force is in the direction opposite to the displacement from equilibrium and has a magnitude proportional to the displacement ($F = -\kappa x$). When this is placed into Newtons' second law ($F = ma$) you get a differential equation as shown here:

$$\begin{aligned} m\frac{d^2x(t)}{dt^2} = ma = F &= -\kappa x(t) \\ \frac{d^2x(t)}{dt^2} + \left(\frac{\kappa}{m}\right) x(t) = \frac{d^2x(t)}{dt^2} + \omega^2 x(t) &= 0 \end{aligned} \tag{6}$$

Equation (6) is sometimes called the simple harmonic oscillator (SHO) equation. The SHO, as you recall, oscillates with frequency $\omega = \sqrt{(\kappa/m)}$. In the parlance of differential equations, it is a *linear*, *second-order*, *homogeneous*, *ordinary differential equation* with *constant coefficients.* It is

- a differential equation because there are derivatives in it,
- ordinary because there are no partial derivatives in it (more on this later),
- second-order because its highest derivative with respect to the independent variable t is of second-order,
- homogeneous because the right-hand-side of the equation is zero which means physically that there are no applied forces, and, finally,
- it has constant coefficients because the terms that multiply the functions of x are constant – in this case it's $\omega^2 = \kappa/m$.

Because this is a linear ODE, if $x_1(t)$ and $x_2(t)$ are solutions, so is $ax_1(t) + bx_2(t)$ where a, b are arbitrary constants. Because it is a second-order ODE, there are two and only two independent solutions. Also within the parlance of differential equations, equations like equation (6) are called *Helmholtz equations.* Ordinary differential equations are often called ODEs.

Helmholtz equations, like the SHO-equation, are particularly easy to solve. The trial solution can be written in a variety of equivalent ways, one of which is:

$$x(t) = A\cos\omega t + B\sin\omega t, \tag{7}$$

where $\omega^2 = \kappa/m$ and A and B are arbitrary constants that depend on the initial conditions – that is on how the oscillator has been set into motion (drag and let go or a kick, for example). Note that there are two independent solutions ($\cos\omega t$, $\sin\omega t$) whose linear combination is also a solution. We can show that equation (7) is a solution to equation (6) by direct substitution:

$$\begin{aligned} \frac{dx(t)}{dt} &= -A\omega\sin\omega t + B\omega\cos\omega t \\ \frac{d^2x(t)}{d^2} &= \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = -A\omega^2\cos\omega t - B\omega^2\sin\omega t \end{aligned} \tag{8}$$

Substitution of equations (7) and (8) into (6) establishes the result:

$$\frac{d^2x}{dt^2} + \omega^2 x = \left(-A\omega^2\cos\omega t - B\omega^2\sin\omega\right) + \omega^2\left(A\cos\omega t + B\sin\omega t\right) = 0.$$

The procedure that we followed here is actually similar to how differential equations are solved in practice, you guess a solution and see if it works. The guessed solution is often called the *trial solution* or *ansatz*, which is the fancier German name for it and you can use this to impress your friends who don't know any better. Now that we know the solution to Helmholtz equations like the SHO equation, we have a starting point for trial solutions later on.

3. The Wave Equation: A Simple Partial Differential Equation in x and t

We consider now the wave equation in 1 spatial dimension; i.e., 1-D. The wave equation in 1-D governs wave propagation in a 1-D medium, such as a string or a wire. There are actually two dimensions in the equation, however, one spatial dimension x and one temporal dimension t. For this reason, we have to handle derivatives slightly differently than in the SHO where the only derivatives were temporal. To do so, we introduce the idea of a *partial derivative*, which is the derivative of a function of two or more variables with respect to one of the variables. The partial derivative is denoted by the symbol ∂ to distinguish it from the total derivative, d, that appears in equation (6). For the string, displacement is a function of both space and time, $y(x,t)$. Thus, we can consider both the spatial and temporal derivatives of displacement, $\partial y(x,t)/\partial x$ and $\partial y(x,t)/\partial t$, which have very different physical meanings. The spatial derivative measures the slope of the string at a particular point and the temporal derivative measures the vertical velocity of the string at a point.

The 1-D wave equation for the string is the following *partial differential equation* (PDE):

$$\frac{\partial^2 y(x,t)}{\partial t^2} = c^2\frac{\partial^2 y(x,t)}{\partial x^2} \tag{9}$$

where $y(x,t)$ represents the displacement of the string. What this says is that the vertical acceleration of the string, $\partial^2 y/\partial t^2$, is proportional to the local curvature of the string, $\partial^2 y/\partial x^2$, and the constant of proportionality is just the square of the speed of waves on the string. This is not intuitively obvious, but let's solve it and see that it yields the solutions discussed physically in section 1 above.

Partial differential equations such as equation (9) are usually not solved directly, but are transformed into other equations that can be solved. Usually they are transformed first into a set of ODEs, one for each free variable. For the 1-D wave equation, therefore, we'll expect two equations, one in x and one in t. The method we're going to follow now is called the *method of separation of variables.*

Equation (9) can be separated into these two constitutive equations by using the method of separation of variables in the following way. Let us assume that the solution can be written (as we know it can for a string) in terms of the product of two functions, one in x and the other in t, in the following way:

$$y(x,t) = Y(x)T(t) \tag{10}$$

$Y(x)$ and $T(t)$ are the unknowns we wish to find and equation (10) is a a kind of trial solution and we'll see if it works. To substitute equation (10) into equation (9) we'll first need the space and time derivatives of y:

$$\begin{aligned}
\frac{\partial y(x,t)}{\partial x} &= T(t)\frac{\partial Y(x)}{\partial x} = T(t)\frac{dY(x)}{dx} \\
\frac{\partial^2 y(x,t)}{\partial x^2} &= T(t)\frac{\partial^2 Y(x)}{\partial x^2} = T(t)\frac{d^2 Y(x)}{dx^2} && (11) \\
\frac{\partial y(x,t)}{\partial t} &= Y(x)\frac{\partial T(t)}{\partial t} = Y(x)\frac{dT(t)}{dt} \\
\frac{\partial^2 y(x,t)}{\partial t^2} &= Y(x)\frac{\partial^2 T(t)}{\partial t^2} = Y(x)\frac{d^2 T(t)}{dt^2} && (12)
\end{aligned}$$

Note that we've replaced the partial derivatives on the right-hand side with total derivatives because they are derivatives of functions of a single variable. Substituting equations (11) and (12) into equation (9) we get:

$$\frac{d^2T(t)}{dt^2}Y(x) = c^2\frac{d^2Y(x)}{dx^2}T(t)$$

which upon rearranging yields:

$$\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} = \frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} \tag{13}$$

Note that the left-hand side of equation (13) is just a function of t and the right-hand side is only a function of x.

Now, comes the key step. It's simple, but you have to pay attention. How can a function of t, which in principle could be changing arbitrarily in time, be equal to a function of x that may be changing arbitrarily in space? Well, to make a long story short, the only way is if both sides of equation (13) are equal to the same constant which is called the *separation constant.* For a reason that will become apparent later, let's let that constant be called $-k^2$, so:

$$\begin{aligned}
\frac{1}{c^2}\frac{1}{T(t)}\frac{d^2T(t)}{dt^2} &= -k^2 \\
\frac{1}{Y(x)}\frac{d^2Y(x)}{dx^2} &= -k^2
\end{aligned}$$

which after a little rearranging can be rewritten as:

$$\begin{aligned}
\frac{d^2Y(x)}{dx^2} + k^2Y(x) &= 0 && (14) \\
\frac{d^2T(t)}{dt^2} + c^2k^2T(t) &= 0 \Rightarrow \frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0 && (15)
\end{aligned}$$

where the latter result in equation (15) holds because $\omega = ck$.

Equations (14) and (15) are the two ODEs whose solutions, $Y(x)$ and $T(t)$, can be substituted into equation (10) to give a solution to the PDE, the wave equation given by equation (9).

Comparison of equations (14) and (15) with equation (6) reveals that both of these equations are simply Helmholtz equations, which we know how to solve because of their role in the SHO. Their solutions, therefore, are simply:

$$
\begin{aligned}
Y(x) &= A\cos kx + B\sin kx && (16)\\
T(t) &= C\cos\omega t + D\sin\omega t && (17)
\end{aligned}
$$

where A, B, C, and D are arbitrary constants. You can see why we defined the separation constant as $-k^2$ because doing so yields equation (16) where k plays the role of wavenumber as we have defined it previously.

The boundary conditions allow us to find A as well as k and, hence, ω as we will now show. The initial conditions will specify the products BC and BD which we will not demonstrate here, but the note on p. 4 briefly discusses how to go about this in general.

Now, let's apply the boundary conditions. The string is clamped both at $x = 0$ and $x = a$. The boundary conditions, therefore, are $y(0, t) = y(a, t) = 0$ or equivalently $Y(0) = Y(a) = 0$, so using equations (16) and (17) we see that:

$$
\begin{aligned}
0 &= Y(0) = A\cos(0) + B\sin(0) \Rightarrow A = 0 && (18)\\
0 &= Y(a) = B\sin ka \Rightarrow k = \frac{1}{a}\sin^{-1}(0) \Rightarrow k_n = \frac{n\pi}{a}, && (19)
\end{aligned}
$$

where n is an integer. Remember that the expression $\sin^{-1}(0)$ should be read as the angle(s) at which sine is zero; which is just multiples of π.

We see, therefore, that we've established that there are a countably infinite number of allowable separation constants k indexed by the number n, that we recognize as the mode number or quantum number as discussed above. In section 1, we established that $k_n = n\pi/a$ based on purely physical considerations, here the reasoning was more mathematical but the result is the same. We see now that:

$$
k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{a} \qquad \omega_n = ck_n = \frac{n\pi c}{a}, \tag{20}
$$

which is the same as equations (3) above. You can see through equations (18) and (19) how the boundary conditions determine the frequencies of oscillation in practice.

The final solution $y(x, t)$ is a linear combination of all of the solutions indexed by n:

$$
\begin{aligned}
y(x,t) &= \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} Y_n(x)T_n(t) = \sum_{n=1}^{\infty} B_n \sin k_n x\,(C_n\cos\omega t + D_n\sin\omega t) && (21)\\
&= \sum_{n=1}^{\infty} \sin k_n x\,(A'_n\cos\omega t + B'_n\sin\omega t) && (22)
\end{aligned}
$$

where we recombined the three arbitrary constants into two as follows: $A'_n \equiv B_nC_n$ and $B'_n \equiv B_nD_n$. This reproduces the physically motivated equation (5) above. As before, the initial conditions will determine the coefficients A'_n and B'_n.