# Physics 2140
# Spring 2005
# Forced, Damped Harmonic Oscillator with Fourier Series RHS

Our purpose is to consider the solution of the following ODE:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m}F(t), \tag{1}$$

where

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\omega_n t + \sum_{n=1}^{\infty} b_n \sin\omega_n t. \tag{2}$$

We show here that once you have the Fourier Series of $F(t)$, that is once you know $a_n, b_n$ and $\omega_n$, you can simply write down the solution to equation (1). The reason is that every term in each of the sums in the Fourier Series has the same functional form ($\sin\omega_n t$ or $\cos\omega_n t$). Terms in the sums differ from one another only in the value of the coefficients $a_n$, $b_n$, and $\omega_n$, which are typically simple functions of $n$. If you solve equation (1) for a single term in each of the sums on its right-hand-side (RHS) you can simply write down the entire solution by summing over $n$. This fact is based on the Principle of Superposition that holds for any linear ODE – which equation (1) is. The Principle of Superposition says that if two functions are separately solutions, then their sum is a solution. So if we know the solution with a single term from each of the sums on the RHS of (1), then we know it for the entire sum.

For an underdamped oscillator, the solution of (1) will be the sum of the complementary and particular solutions:

$$\begin{aligned} x(t) &= x_c(t) + x_p(t), && (3)\\ x_c(t) &= Ce^{-\beta t}\cos(\omega_1 t + \psi), && (4)\\ \omega_1 &= \sqrt{\omega_0^2 - \beta^2}, && (5) \end{aligned}$$

where $C$ and $\psi$ are constants that depend on $x_0$ and $v_0$. Thus, since we already know $x_c(t)$, the rest of our work will involve finding $x_p(t)$.

For starters consider the equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m}e^{i\omega t}. \tag{6}$$

Try $x_p = A\exp(i\omega t)$, where $A$ is a complex constant which we wish to find. Substitution into equation (6) yields:

$$\left(-\omega^2 + 2i\beta\omega + \omega_0^2\right)A = \frac{1}{m} \Rightarrow A = \frac{1}{m}\left[\frac{(\omega_0^2 - \omega^2) - 2i\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\right]. \tag{7}$$

Note that $A$ is complex, written in rectangular form. Let's convert it to polar form by finding its modulus $|A|$ and phase $\phi$. Thus,

$$x_p(t) = |A|e^{i\phi}e^{i\omega t} = |A|e^{i(\omega t+\phi)}, \tag{8}$$

$$|A| = \frac{1}{m}\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{-1/2}, \tag{9}$$

$$\phi = \tan^{-1}\left(\frac{-2\beta\omega}{\omega_0^2-\omega^2}\right). \tag{10}$$

Now consider the following two equations:

$$\ddot{x}+2\beta\dot{x}+\omega_0^2x = \frac{a}{m}\cos\omega t = \frac{a}{m}\mathrm{Re}(e^{i\omega t}), \tag{11}$$

$$\ddot{x}+2\beta\dot{x}+\omega_0^2x = \frac{b}{m}\sin\omega t = \frac{b}{m}\mathrm{Im}(e^{i\omega t}). \tag{12}$$

Since $\cos(\omega t)$ and $\mathrm{sine}(\omega t)$ are simply the real and imaginary parts of $\exp(i\omega t)$, respectively, the solutions to equations (11) and (12) are very simply related to the solution of (6). In fact, they are simply the real and imaginary parts of the solution given by equation (8):

$$x_p(t) = |A|\cos(\omega t+\phi_n), \qquad \text{for cosine RHS,} \tag{13}$$

$$x_p(t) = |A|\sin(\omega t+\phi_n), \qquad \text{for sine RHS,} \tag{14}$$

where $|A|$ and $\phi$ are again given by equations (9) and (10).

Finally, if $F(t)$ is expressed in a Fourier Series as in equation (2), then we simply sum over all components of each sum. The solution is then:

$$x_p(t) = \frac{a_0}{2m\omega_0^2}+\sum_{n=1}^{\infty}A_n\cos(\omega_nt+\phi)+\sum_{n=1}^{\infty}B_n\sin(\omega_nt+\phi), \tag{15}$$

$$A_n = \frac{a_n}{m}\left[(\omega_0^2-\omega_n^2)^2+4\beta^2\omega_n^2\right]^{-1/2}, \tag{16}$$

$$B_n = \frac{b_n}{m}\left[(\omega_0^2-\omega_n^2)^2+4\beta^2\omega_n^2\right]^{-1/2}, \tag{17}$$

$$\phi_n = \tan^{-1}\left(\frac{-2\beta\omega_n}{\omega_0^2-\omega_n^2}\right). \tag{18}$$

Note that the first term in equation (15) comes from the constant $a_0/2$ in the Fourier Series.

Equations (15) - (18) give the results we seek. The particular solution to any RHS that is expressible as a Fourier Series is given by these equations. It is only necessary to find $a_n, b_n$, and $\phi_n$, which come directly from the Fourier Series. The key to finding the solution of equation (1), therefore, is simply to find the Fourier Series of $F(t)$, equation (2).