Physics 2140
Methods in Theoretical Physics
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The Wave Equation in Two Spatial Dimensions

In more than one spatial dimension, spatial derivatives are replaced by a vector operator called the gradient ∇ that operates on a scalar field and whose form will depend on the system of coordinates: rectangular (Cartesian), cylindrical, spherical, and so forth. In 2-D Cartesian coordinates, the gradient is defined as:

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}, \tag{1}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are Cartesian unit vectors in the orthogonal x and y directions. Second spatial derivatives will be replaced with the Laplacian operator, $\nabla^2 = \nabla \cdot \nabla$, which operates on a scalar field and returns a scalar. In 2-D Cartesian coordinates the Laplacian is given by:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2}$$

In other coordinates systems, the gradient and Laplacian operators will be more complicated, but that will not be discussed here.

1. The 2-D Wave Equation in Cartesian Coordinates

When we considered the 1-D wave equation, we represented the field variable, displacement, with $y(x,t)$. In 2-D, we'll represent displacement with $u(x,y,t)$, because now y is one of the two variables that represent spatial dimensions. Replacing the second spatial derivative with the 2-D Laplacian, the 2-D wave equation can be written:

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{3}$$

$$\tag{4}$$

which in 2-D Cartesian coordinates becomes:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{5}$$

We will consider a 2-D rectangular membrane or drum-head (the 2-D analog of a string) stretched between $0 \le x \le a$ and $0 \le y \le b$. The edges of the membrane will be considered to be clamped, so the boundary conditions are:

$$u(0,y,t) = u(a,y,t) = 0 \tag{6}$$

$$u(x,0,t) = u(x,b,t) = 0. \tag{7}$$

These are homogeneous, Dirichlet boundary conditions.

We now want to apply separation of variables to equation (5). We'll do it in two stages; first by separating the temporal from the spatial parts of the equation and then by separating the two spatial parts. We'll end up with three ODE's and two separation constants by the time we're done. First, substitute

$$u(x,y,t) = U(x,y)T(t) \tag{8}$$

into equation (5), divide by $U(x,y)T(t)$, and replace the temporal partial derivative by the:

$$\frac{1}{c^2T(t)}\frac{d^2T}{dt^2} = \frac{1}{U(x,y)}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) = -\beta^2. \tag{9}$$

We introduced the separation constant $-\beta^2$ on the right because we have a function of the spatial variables equal to a function of time which requires that both sides equal a constant, the first separation constant $-\beta^2$. This yields two ODEs:

$$\frac{d^2T}{dt^2} + c^2\beta^2 T(t) = 0, \tag{10}$$

$$\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \beta^2 U(x,y) = 0. \tag{11}$$

Equation (10) is the temporal equation, and we're done with it. It has solution:

$$T(t) = A\cos\omega t + B\sin\omega t, \qquad \text{where} \qquad \omega = c\beta. \tag{12}$$

The spatial equation, equation (11), requires further separation as there remain two spatial variables and the derivatives are partial derivatives. Let's apply separation of variables again by letting:

$$U(x,y) = X(x)Y(y). \tag{13}$$

Substituting this into equation (11), dividing by XY and replacing the partial derivatives by total derivatives wet get:

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y}{dy^2} + \beta^2 = 0. \tag{14}$$

This can be rewritten as:

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} + \beta^2 = -\frac{1}{Y(y)}\frac{d^2Y}{dy^2} = q^2, \tag{15}$$

where, because we have a function of x equal to a function of y, we have introduced the second separation constant, q^2. From this we get the following two spatial ODE's:

$$\frac{d^2X}{dx^2} + (\beta^2 - q^2)X(x) = \frac{d^2X}{dx^2} + p^2X(x) = 0, \tag{16}$$

$$\frac{d^2Y}{dy^2} + q^2Y(y) = 0, \tag{17}$$

where in equation (16) we have defined $p^2 \equiv \beta^2 - q^2$, or:

$$\beta^2 = p^2 + q^2. \tag{18}$$

We'll come back to this equation later. It says that the first separation constant depends on the spatial separation constants and this will be important in the solution of the temporal equation, equation (12), as frequency depends on β.

Solving the spatial ODE's we get:

$$X(x) = C\cos px + D\sin px, \tag{19}$$

$$Y(y) = E\cos qy + F\sin qy. \tag{20}$$

Now, let's apply the boundary conditions, which in terms of $X(x)$ and $Y(y)$ can be rewritten:

$$\begin{aligned} X(0) = X(a) &= 0 \qquad (21)\\ Y(0) = Y(b) &= 0. \qquad (22)\end{aligned}$$

Applying equation (21) to equation (19) and then equation (22) the the result:

$$\begin{aligned} 0 &= X(0) = C \rightarrow X(x) = D \sin px \qquad (23)\\ 0 &= X(a) = D \sin pa \rightarrow p_n = \frac{1}{a}\sin^{-1}(0) = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \ldots. \qquad (24)\end{aligned}$$

Thus, we get the solution for $X(x)$. Following exactly the same procedure we find

$$q_n = \frac{1}{b}\sin^{-1}(0) = \frac{m\pi}{b}, \qquad m = 1, 2, 3, \ldots. \qquad (25)$$

and the solution for $Y(y)$:

$$\begin{aligned} X_n(x) &= D_n \sin\left(\frac{n\pi x}{a}\right) \qquad n = 1, 2, 3, \ldots \qquad (26)\\ Y_m(y) &= F_m \sin\left(\frac{m\pi y}{b}\right) \qquad m = 1, 2, 3, \ldots \qquad (27)\end{aligned}$$

Note that we introduced two quantum numbers here, n and m.

Before moving on to write the general solution, recall that $\beta^2 = p^2 + q^2$. Because p depends on n and q depends on m, β will depend on both quantum numbers as follows:

$$\beta_{nm}^2 = p_n^2 + q_m^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right) \qquad (28)$$

Different modes of oscillation in 2-D are identified by a pair of quantum numbers (n, m).

This discretization of β by the boundary conditions also discretizes the temporal solution, equation (12):

$$T_{nm}(t) = A_{nm}\cos\omega_{nm}t + B_{nm}\sin\omega_{nm}t, \quad \text{where} \quad \omega_{nm} = c\beta_{nm} = c\pi\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}. \qquad (29)$$

To get the general solution, we put equations (26) and (27) together with equation (29). In doing so, we let $A_{nm}D_n = a_{nm}$ and $B_{nm}F_m = b_{nm}$. The result is a pair of sums over all possible n and m:

$$\begin{aligned} u(x, y, t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} u_{mn}(x, y.t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T_{nm}(t)X_n(x)Y_m(y)\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}\cos\omega_{nm}t + b_{nm}\sin\omega_{nm}t)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right), \qquad (30)\end{aligned}$$

where

$$\omega_{nm} = c\beta_{nm} = c\pi\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}. \qquad (31)$$

Equation (30) is the general solution to the 2-D wave equation in Cartesian coordinates with clamped boundaries. Like the string equation, it is expressed as a sum over normal modes, $u_n m(x, y, t)$. Equation (30) gives us the allowed frequencies of oscillation for each mode (n, m).

It remains to apply the initial conditions. To evaluate the arbitrary constants a_{nm} and b_{nm} in equation (30), we will need both the initial displacement and velocity of the membrane. The process will depend on using the orthogonality of the two sets of functions $\{\sin(n\pi x/a)\}$ and $\{\sin(m\pi y/b)\}$.

First, suppose that the initial displacement of the membrane can be described by:

$$u(x, y, 0) = f(x, y). \tag{32}$$

Setting $t = 0$ in equation (30) and applying equation (32) gives:

$$u(x, y, 0) = f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right). \tag{33}$$

This is a double Fourier-series. To evaluate the coefficient a_{nm}, multiply both sides of equation (33) by $\sin(n'\pi x/a)$ and $\sin(m'\pi y/b)$, and integrate over the area of the membrane. Using the orthogonality property of the sinusoids we will find that:

$$a_{nm} = \frac{2}{b}\frac{2}{a} \int_0^a \int_0^b f(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dydx. \tag{34}$$

Similarly, to evaluate the coefficients b_{nm}, we use the initial velocity field of the membrane:

$$\dot{u}(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{nm} \left(-a_{nm} \sin \omega_{nm} t + b_{nm} \cos \omega_{nm} t\right) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right). \tag{35}$$

If the initial velocity field is $\dot{u}(x, y, 0) = g(x, y)$, then following the same procedure we took to find the a_{nm}:

$$\begin{aligned} \dot{u}(x, y, 0) &= g(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \omega_{nm} b_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) & (36) \\ b_{nm} &= \frac{1}{\omega_{nm}} \frac{2}{b}\frac{2}{a} \int_0^a \int_0^b g(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dydx, & (37) \end{aligned}$$

where, of course, ωnm is given by equation (31).