

Homework 9
Physics 2140
Methods in Theoretical Physics
Due: Wednesday April 6, 2005

Homework Assignment

1. Boas 8.5.

4. $y'' + 2y' + 2y = 0$. Trying $y(x) = e^{sx}$ you get the auxiliary equation:

$$s^2 + 2s + 2 = 0, \quad (1)$$

which has complex roots $s_1 = -1 - i$ and $s_2 = -1 + i$. Thus the solution is

$$y(x) = Ae^{(-1-i)x} + Be^{(-1+i)x} \quad (2)$$

$$= e^{-x} (Ae^{-ix} + Be^{ix}). \quad (3)$$

We're not told that $y(x)$ is real. If it were, then $B = A^*$.

13. Let $y_1(x) = e^{-x}$ and $y_2 = e^{-4x}$.

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-4x} \\ -e^{-x} & -4e^{-4x} \end{vmatrix} = -4e^{-5x} + e^{-5x} = -3e^{-5x} \neq 0. \quad (4)$$

Thus the set $\{e^{-x}, e^{-4x}\}$ is linearly independent.

24. $y''' + y = 0$. Trying $y(x) = e^{sx}$ you get the auxiliary equation:

$$s^3 + 1 = (s + 1)(s^2 - s + 1) = 0, \quad (5)$$

which has roots equal to $s_1 = -1, s_2 = \frac{1}{2}(1 + i\sqrt{3}), s_3 = \frac{1}{2}(1 - i\sqrt{3})$. Thus the solution to the differential equation is

$$y(x) = Ae^{-x} + Be^{(1+i\sqrt{3})x/2} + Ce^{(1-i\sqrt{3})x/2}. \quad (6)$$

This can be written in a number of ways, but this is one is the easiest to find.

34. We derived the pendulum equation in class earlier in the semester. Consider Figure 1. If we work in a coordinate system attached to the pendulum bob of mass m and if the direction of motion of the bob is called s , then the component of gravity along s is just $-mg \sin \theta$. So using Newton's second law:

$$F = m\ddot{s} = -mg \sin \theta \rightarrow \ddot{s} + g \sin \theta = 0. \quad (7)$$

An arc of the circle traveled by the pendulum bob, will be just $s = L\theta$, so $\ddot{s} = L\ddot{\theta}$. Substituting into equation (7) gives the pendulum equation:

$$\ddot{\theta} + \frac{g}{L} \sin \theta = 0. \quad (8)$$

In the event of small amplitude oscillations, $\sin \theta \sim \theta$, and you get the small amplitude equation:

$$\ddot{\theta} + \frac{g}{L} \theta = 0. \quad (9)$$

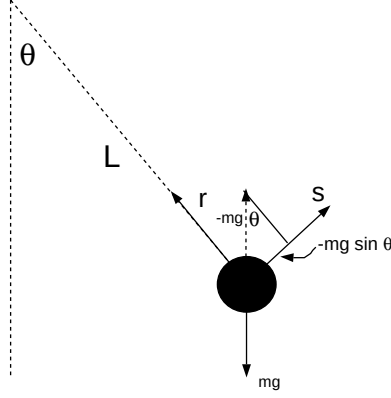


Figure 1:

This is just a second order ODE with constant coefficients, which we can solve using the trial solution $\theta(t) = e^{st}$. Substituting into equation (9) we get the auxiliary equation $s^2 + (g/L) = 0$ whose roots are $s_{1,2} = \pm i\sqrt{g/L} = \pm\omega_0$ if we let $\omega_0 = \sqrt{g/L}$. Thus, the general solution to equation (9) will be:

$$\theta(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}. \quad (10)$$

Because θ is a real quantity, $B = A^*$, so (10) can be rewritten as follows if we let $A = a + ib$ where a, b are real:

$$\theta(t) = Ae^{i\omega_0 t} + A^*e^{-i\omega_0 t} = Ae^{i\omega_0 t} + (Ae^{-i\omega_0 t})^* = 2\text{Re}[Ae^{i\omega_0 t}] \quad (11)$$

$$= 2(a \cos(\omega_0 t) - b \sin(\omega_0 t)) = C \cos(\omega_0 t) + D \sin(\omega_0 t), \quad (12)$$

where C, D are two new real constants.

Applying the initial condition to equation (12) that $\theta(0) = \theta_0$ we find that $C = \theta_0$. Finally, applying the condition that $\dot{\theta}(0) = 0$ we see that $D = 0$. The final solution that satisfies these initial conditions is just the equation for simple harmonic motion:

$$\theta(t) = \theta_0 \cos(\omega_0 t). \quad (13)$$

- 35.** We want to find the time variable distance of the particle from the center of the earth, $r(t)$, and we're given that $ma = m\ddot{r} = F = -mgr/R$ toward the earth's center where R is the earth's radius ($r < R$). Thus

$$\ddot{r} + \frac{g}{R}r = 0. \quad (14)$$

Try $r(t) = e^{st}$ and the roots of the auxiliary equation ($s^2 + g/R = 0$) are:

$$s_1 = i\sqrt{\frac{g}{R}} \quad s_2 = -i\sqrt{\frac{g}{R}}, \quad (15)$$

so

$$r(t) = Ae^{i\sqrt{g/R}t} + Be^{-i\sqrt{g/R}t}. \quad (16)$$

This is just the equation for simple harmonic motion where frequency $\omega = \sqrt{g/R}$ and period $T = 2\pi/\omega = 2\pi\sqrt{R/g} = 5066 \text{ s} \sim 84.4 \text{ min}$. That is, about 42 min through the earth and 42 min back.

2. Boas 8.6. These are all simple inhomogeneous ODE's. Do these problems in two steps. First, solve the homogeneous equation to find the complementary solution, $y_c(x)$. Second, use the Method of Undetermined Coefficients (MUC) to find the particular solution, $y_p(x)$. If there are multiple terms on the right-hand-side (RHS) of the ODE, then you invoke the Principal of Superposition and apply the MUC to a set of inhomogeneous ODE's each with only a single term on the RHS. The particular solution to the original ODE is then the sum of the particular solutions to each of these simpler inhomogeneous ODE's.

1. $y'' - 4y = 10$. This is the simplest inhomogeneous ODE since it has a constant RHS. The roots of the homogeneous equation are: $s_1 = 2, s_2 = -2$, so $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. With a constant RHS, try $y_p = A$. Substituting into the ODE you get $0 - 4A = 10 \rightarrow A = -5/2$. Thus, the general solution is $y(x) = y_c(x) + y_p(x) = c_1 e^{2x} + c_2 e^{-2x} - 5/2$.

6. $y'' + 6y' + 9y = 12e^{-x}$. The roots of the auxiliary equation are $s_1 = -3, s_2 = -3$, which are degenerate. The complementary solution is, therefore, of the form $y_c(x) = (Ax + B)e^{sx}$, where s is the root of the auxiliary equation. Now turning to the particular solution, using the MUC the general form for the RHS is $e^{cx} P_n(x)$ where $P_n(x)$ is a polynomial of degree n . Here, $c = -1, P_n(x) = 12, n = 0$. Note that $s_1 = s_2 \neq c$, so we try a particular solution of the form $y_p(x) = De^{-x}$. Since $y_p'(x) = -De^{-x}, y_p''(x) = De^{-x}$, substitution into the original ODE gives:

$$(1 - 6 + 9)De^{-x} = 12e^{-x} \rightarrow D = 3, \quad (17)$$

and the general solution is

$$y(x) = y_c(x) + y_p(x) \quad (18)$$

$$= (Ax + B)e^{-3x} + 3e^{-x}. \quad (19)$$

17. $y'' + 16y = 16 \cos 4x = 16 \operatorname{Re}(e^{i4x})$. Note that from the outset we have written cosine on the RHS as the real part of an exponential because the MUC is for RHS's with exponentials in them. What we will then do is find a particular solution where the real part comes from the cosine component of e^{i4x} and the imaginary part comes from the sine component. Thus since we are interested here in a cosine RHS we will retain only the real part of the particular solution. Alternately, we could have noted that $\cos$ on the RHS can be broken into a sum of two exponentials, $\cos(4x) = (\exp(4ix) + \exp(-4ix))/2$, and used the Principal of Superposition. The way we choose to do the problem here is a little easier.

The roots of the homogeneous equation are: $s_1 = 4i, s_2 = -4i$ and the complementary solution is $y_c(x) = c_1 e^{i4x} + c_2 e^{-i4x}$. Our RHS here is $16e^{i4x}$, so $c = 4i, P_n(x) = 1, n = 0$. We see that $s_1 \neq s_2$, but $s_1 = c$ so by the MUC we try a particular solution of the form $y_p(x) = Axe^{4ix}$. Thus

$$y'' + 16y = (-16x + 8i + 16x)Ae^{i4x} = 16e^{i4x} \rightarrow A = -2i, \quad (20)$$

$$y_p(x) = -2ixe^{4ix} = -2ix(\cos 4x + i \sin 4x) = 2x \sin 4x - i2x \cos 4x, \quad (21)$$

$$y(x) = y_c(x) + \operatorname{Re}(y_p(x)) = c_1 e^{i4x} + c_2 e^{-i4x} + 2x \sin 4x. \quad (22)$$

35. $y'' - y = \sinh x = (e^x - e^{-x})/2$. Note that we've written the RHS from the outset as the difference between two exponentials so that we can apply the MUC. The roots of the homogeneous equation are $s_1 = 1, s_2 = -1$ so the complementary solution is $y_c(x) = c_1 e^x + c_2 e^{-x}$. To find the particular solution, we consider two equations independently:

$$y'' - y = \frac{1}{2}e^x \quad (23)$$

$$y'' - y = -\frac{1}{2}e^{-x} \quad (24)$$

Considering equation (23), the RHS has $c = 1, P_n(x) = 1, n = 0$ so we see that $s_1 \neq s_2$, but $s_1 = c$. Thus we try $y_{p1}(x) = Axe^x$. Substitution into equation (23) yields $A = 1/4$, hence $y_{p1}(x) = \frac{1}{4}xe^x$. Similarly, considering equation (24), the RHS has $c = -1, P_n(x) = 1, n = 0$ so we see that $s_1 \neq s_2$, but $s_2 = c$. Thus again we try $y_{p2}(x) = Bxe^{-x}$. Substitution into equation (24) yields $B = 1/4$, hence $y_{p2}(x) = \frac{1}{4}xe^{-x}$.

The general solution then is:

$$y(x) = y_c(x) + y_{p1}(x) + y_{p2}(x) = c_1e^x + c_2e^{-x} + \frac{1}{4}xe^x + \frac{1}{4}xe^{-x} \quad (25)$$

$$= c_1e^x + c_2e^{-x} + \frac{1}{4}x(e^x + e^{-x}) = c_1e^x + c_2e^{-x} + \frac{1}{2}x \cosh x. \quad (26)$$

3. The undamped, unforced simple harmonic oscillator equation is:

$$\ddot{x} + \omega^2 x = 0. \quad (27)$$

Solutions to this equation are written in a variety of different forms, including:

$$x(t) = A \sin(\omega t + \phi) \quad (28)$$

$$x(t) = B \cos(\omega t + \psi) \quad (29)$$

$$x(t) = C \cos(\omega t) + D \sin(\omega t) \quad (30)$$

$$x(t) = Ee^{i\omega t} + E^*e^{-i\omega t}, \quad (31)$$

where A, B, C, D, ϕ, ψ are real, E is complex, and $*$ denotes complex conjugation.

a. What is the relation between (A, ϕ) and (B, ψ) in equations (28) and (29)?

$$A = B, \psi = \phi + \pi/2.$$

b. What is the relation between (A, ϕ) and (C, D) in equations (28) and (30)?

$$A \sin(\omega t + \phi) = A \sin \omega t \cos \phi + A \cos \omega t \sin \phi \quad (32)$$

$$= (A \cos \phi) \sin \omega t + (A \sin \phi) \cos \omega t \quad (33)$$

$$= C \cos \omega t + D \sin \omega t. \quad (34)$$

$$\rightarrow C = A \sin \phi, D = A \cos \phi. \quad (35)$$

c. What is the relation between (B, ψ) and $(\text{Re } E, \text{Im } E)$ in equations (29) and (31), where $\text{Re } E$ means the real part of E and $\text{Im } E$ means imaginary part of E ?

$$B \cos(\omega t + \psi) = B \left[\frac{e^{i(\omega t + \psi)} + e^{-i(\omega t + \psi)}}{2} \right] \quad (36)$$

$$= \left(\frac{B}{2} e^{i\psi} \right) e^{i\omega t} + \left(\frac{B}{2} e^{-i\psi} \right) e^{-i\omega t} \quad (37)$$

$$= Ee^{i\omega t} + E^*e^{-i\omega t}. \quad (38)$$

$$\rightarrow E = \frac{B}{2} e^{i\psi}, E^* = \frac{B}{2} e^{-i\psi} \quad (39)$$

$$\rightarrow \text{Re } E = \frac{B}{2} \cos \psi, \text{Im } E = \frac{B}{2} \sin \psi \quad (40)$$

d. If we replace E^* in equation (31) with another completely arbitrary complex constant F :

$$x(t) = Ee^{i\omega t} + Fe^{-i\omega t}, \quad (41)$$

the resulting equation is still a solution to equation (27). Why do we not consider equation (41) as a solution to the undamped, unforced simple harmonic oscillator equation, but rather use a specific form of equation (41) in which $F = E^*$ as the solution?

If $F \neq E^*$, then $x(t)$ would be complex, which is not physically realistic.

4. The unforced, damped harmonic oscillator equation:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \quad (42)$$

The following is a solution of this equation when it is underdamped ($\omega_0^2 > \beta^2$):

$$x(t) = Ae^{-\beta t} \cos(\omega_1 t + \phi), \quad (43)$$

where $\omega_1^2 = \omega_0^2 - \beta^2$.

a. Find A and ϕ in terms of displacement and velocity at time $t = 0$, x_0 and v_0 .

$$x_0 = A \cos \phi \quad (44)$$

$$v(t) = Ae^{-\beta t} [-\omega_1 \sin(\omega_1 t + \phi) - \beta \cos(\omega_1 t)] \quad (45)$$

$$\rightarrow v_0 = -\omega_1 A \sin \phi - \beta(A \cos \phi) = -\omega_1 A \sin \phi - \beta x_0 \quad (46)$$

$$\rightarrow A \sin \phi = \frac{v_0 + \beta x_0}{-\omega_1}. \quad (47)$$

Taking the sum of the squares of equations (44) and (47) gives:

$$A^2 = x_0^2 + \frac{(v_0 + \beta x_0)^2}{\omega_1^2} \rightarrow A = \frac{1}{\omega_1} [x_0^2 \omega_1^2 + (v_0 + \beta x_0)^2]^{1/2} \quad (48)$$

Taking the ratio of equations (44) and (47) gives:

$$\phi = \tan^{-1} \left(\frac{v_0 + \beta x_0}{-\omega_1 x_0} \right). \quad (49)$$

b. Show that in the absence of damping ($\beta = 0$), if $v_0 = 0$, then $A = x_0$ and $\phi = 0$.

There's not much to show here. From equation (48), $A = x_0$ follows immediately. In equation (49), if $v_0 = \beta = 0$, then the numerator of the fraction is zero, and $\phi = \tan^{-1}(0) = 0$.

c. Again assume that $v_0 = 0$, but now let there be damping ($\beta \neq 0$). Show that $A > x_0$. How can this be? How can the constant A be greater than the initial displacement? Does this make sense?

$$A = \frac{1}{\omega_1} (x_0^2 \omega_1^2 + \beta^2 x_0^2)^{1/2} = x_0 \left(\frac{\omega_1^2 + \beta^2}{\omega_1^2} \right)^{1/2} > x_0, \quad (50)$$

when $v_0 = 0$. A should not be equal to x_0 . Rather $A \cos(\phi)$ should equal x_0 . $A > x_0$ since $\cos(\phi) < 1$ when $\phi \neq 0$. A is just larger than x_0 to compensate for $\cos \phi$ being less than 1. It makes sense.