

# Homework 8

## Physics 2140

### Methods in Theoretical Physics

#### Due: Monday March 14, 2005

**Reading Assignment:** Thornton and Marion, continue Ch. 2.

1. Thornton and Marion 2-32.

The force on the mass on the incline is

$$m\ddot{x} = mg - 2mg \sin \theta - 2\mu mg \cos \theta \quad (1)$$

To move with a constant speed,  $\ddot{x} = 0$ . This will occur at angle  $\theta_0$ . Thus,

$$0 = 1 - 2 \sin \theta_0 + 2\mu \cos \theta_0 \quad (2)$$

$$\frac{1}{2} = \sin \theta_0 + \mu \cos \theta_0 \quad (3)$$

$$\frac{1}{2} = \sin \theta_0 + \mu \left(1 - \sin^2 \theta_0\right)^{1/2} \quad (4)$$

$$(5)$$

Now, isolating the square root, squaring and rearranging:

$$\left(1 + \mu^2\right) \sin^2 \theta_0 - \sin \theta_0 + \left(\frac{1}{4} - \mu^2\right) = 0 \quad (6)$$

$$\sin \theta_0 = \frac{1 \pm \mu \sqrt{3 + \mu^2}}{2(1 + \mu^2)} \quad (7)$$

2. Thornton and Marion 2-39.

**(a)**

$$F = m \frac{dv}{dt} = -\alpha e^{\beta v}, \quad (8)$$

$$\frac{dv}{dt} = -\frac{\alpha}{m} e^{\beta v}, \quad (9)$$

$$\int e^{-\beta v} dv = -\frac{\alpha}{m} \int dt, \quad (10)$$

$$-\beta^{-1} e^{-\beta v} = -\frac{\alpha}{m} t + c. \quad (11)$$

We've done the problem with indefinite integrals, so to find the integration constant  $c$  we must apply the initial condition that  $v = v_0$  at  $t = 0$ . (Alternately we could've applied the appropriate limits to the integral and done the definite integral.) Applying the initial condition implies that

$$c = -\beta^{-1} e^{-\beta v_0} \rightarrow -\beta^{-1} \left(e^{-\beta v} - e^{-\beta v_0}\right) = -\frac{\alpha}{m} t. \quad (12)$$

Solving for  $v$  gives:

$$v(t) = -\beta^{-1} \ln \left[ \frac{\alpha \beta t}{m} + e^{-\beta v_0} \right]. \quad (13)$$

(b) Solve for  $t$  when  $v = 0$ :

$$\frac{\alpha\beta t}{m} + e^{-\beta v_0} = 1 \rightarrow t = \frac{m}{\alpha\beta} (1 - e^{-\beta v_0}). \quad (14)$$

(c) Start with equation (13):

$$dx = -\beta^{-1} \ln \left( \frac{\alpha\beta t}{m} + e^{-\beta v_0} \right) dt. \quad (15)$$

Using the fact that (from a table of integrals or mathematica):

$$\int \ln(at + b) dx = \frac{at + b}{a} \ln(at + b) - t, \quad (16)$$

we get:

$$x + c = -\frac{1}{\beta} \left[ \frac{\left( \frac{\alpha\beta t}{m} + e^{-\beta v_0} \right) \ln \left( \frac{\alpha\beta t}{m} + e^{-\beta v_0} \right)}{\alpha\beta/m} - t \right] \quad (17)$$

Applying the initial condition that  $x = 0$  at  $t = 0$  to find  $c$  we get:

$$c = \frac{v_0 m}{\alpha\beta} e^{-\beta v_0}, \quad (18)$$

$$x(t) = -\frac{mv_0}{\alpha\beta} e^{-\beta v_0} + \frac{t}{\beta} - \frac{m}{\alpha\beta^2} \left( \frac{\alpha\beta t}{m} + e^{-\beta v_0} \right) \ln \left( \frac{\alpha\beta t}{m} + e^{-\beta v_0} \right). \quad (19)$$

Substituting the time required to stop (eqn. 14), gives the distance required to stop:

$$x = \frac{m}{\alpha\beta} \left[ \frac{1}{\beta} - e^{-\beta v_0} \left( v_0 + \frac{1}{\beta} \right) \right]. \quad (20)$$

### 3. Thornton and Marion 2-54.

(a) The terminal speed means that the final speed is constant, or the net force is zero:  $mg = kmv \rightarrow v = g/k = 10/.01 \sim 980$  m/s.

(b)

$$\frac{dv}{dt} = \frac{F}{m} = -(g)kv \rightarrow \frac{dx}{v} = dt = -\frac{dv}{g + kv} \quad (21)$$

$$\int_0^x dx' = -\int_{v_0}^0 \frac{v dv}{g + kv} \rightarrow x = \frac{v_0}{k} + \frac{g}{k^2} \ln \frac{g}{g + kv_0} \sim 680 \text{ m}. \quad (22)$$

### Energy Problems:

#### 4. Thornton and Marion 2-25 (a) - (c).

(a) The track counters the force of gravity and, importantly, centripetal acceleration. In polar coordinates centered at the center of the circular arc around point A, the radial component of Newton's second law tells us that:

$$ma_r = m(\ddot{r} - r\dot{\theta}^2) = mg - N. \quad (23)$$

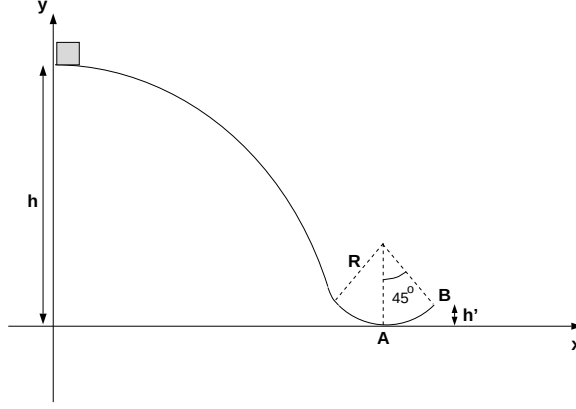


Figure 1: Geometry for problem #4.

Note the signs on the right hand side of equation (23). Gravity points in the positive  $r$  direction and is, therefore, positive and the normal force  $N$  points in the negative radial direction and is negative. Near point A, the block moves along the arc of a circle so  $\ddot{r} = 0$ ,  $r = R$ , and  $\dot{\theta} = \omega = v/R$ . Thus,

$$-\frac{mv^2}{R} = mg - N \Rightarrow N = mg + \frac{mv^2}{R}. \quad (24)$$

To finish the derivation, we need to express  $v$  in terms of  $h$  by using the conservation of energy:

$$E_{top} = U_{top} = mgh = E_A = U_A + K_A = 0 + \frac{1}{2}mv^2 \Rightarrow v^2 = 2gh. \quad (25)$$

From equations (24) and (25) we see that:

$$N = mg + \frac{m}{R}2gh = mg \left(1 + \frac{2h}{R}\right). \quad (26)$$

(b) Similarly, at point B

$$N = mg \cos \theta + \frac{mv^2}{R} = \frac{mg}{\sqrt{2}} + \frac{mv^2}{R} \quad (27)$$

To find  $N$  we need to find  $v$  using conservation of energy again:

$$E = mgh = \frac{1}{2}mv^2 + mgh' = \frac{1}{2}mv^2 + mg \left(R - \frac{R}{\sqrt{2}}\right) \Rightarrow v^2 = 2g \left[h - R \left(1 - \frac{1}{\sqrt{2}}\right)\right]. \quad (28)$$

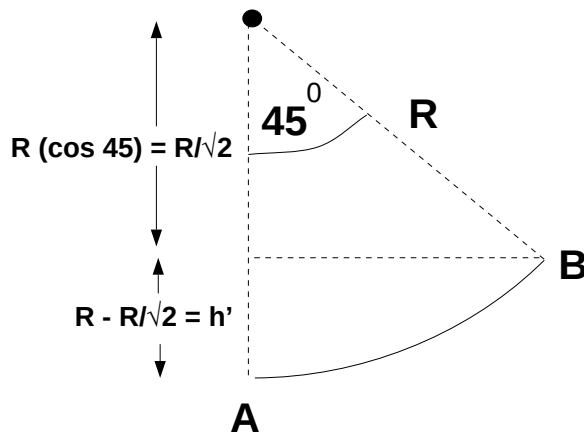
The following figure helps to see that  $h' = R - R/\sqrt{2}$ .

Thus,

$$N = \frac{mg}{\sqrt{2}} + \frac{2gm}{R} \left[h - R \left(1 - \frac{1}{\sqrt{2}}\right)\right] = mg \left[\frac{2h}{r} + \left(\frac{3}{\sqrt{2}} - 2\right)\right] \quad (29)$$

(c) From equation (28) we see that

$$v = \sqrt{2g \left[h - R \left(1 - \frac{1}{\sqrt{2}}\right)\right]}. \quad (30)$$



5. Thornton and Marion 2-43.

In discussing the motion of the particle, clearly indicate at what energies the particle is free (or unbound), trapped, and where the stable and unstable equilibrium points are. Please make sure to plot – although Marion does not ask you to do so.

$$F = -kx + kx^3/\alpha^2 \rightarrow U(x) = -\int F(x)dx = \frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2}.$$

The figure on the last page of the homework solutions presents a computer plot of  $U(x)$ . You don't need to plot this potential with a computer for this assignment, however. Merely note that near  $x = 0$ ,  $U(x)$  behaves like the parabola  $kx^2/2$ , which is concave up and possesses a minimum at the origin. For very large values of  $\pm x$ , it behaves like  $-\frac{1}{4}k\frac{x^4}{\alpha^2}$ , which goes to negative infinity. Thus,  $U(x)$  displays two maxima, symmetrically distributed on either side of the origin. The location of the three extrema, are roots of the following equation:

$$\frac{dU}{dx} = 0 \rightarrow 0 = kx - \frac{k}{\alpha^2}x^3 \rightarrow x = 0, \pm\alpha. \quad (31)$$

Thus, in addition to the stable equilibrium point at  $x = 0$ , the two unstable equilibria are at  $x = \pm\alpha$ . To check for stability/instability, note that at  $x = 0$ ,  $d^2U/dx^2 = k > 0$ , so that  $x = 0$  is a stable equilibrium point, but at  $x = \pm\alpha$ ,  $d^2U/dx^2 = -2k < 0$ , so  $x = \pm\alpha$  are points of unstable equilibrium. Note that at  $x = \pm\alpha$  (points 4 and 7) on the graph,  $U(\pm\alpha) = E_1 = k\alpha^2/4$ . Thus,  $k\alpha^2/4$  is the potential energy of the unstable equilibrium points. Note that  $U = 0$  at  $x = \pm\sqrt{2}\alpha$  (points 2 and 9 on the graph) and at the origin.

If  $E_0$  is everywhere greater than  $k\alpha^2/4$ , then the particle is free.

If  $E_1 = k\alpha^2/4$ , then the particle is unbound everywhere except at  $x = \pm\alpha$  (points 4 and 7), where it is in unstable equilibrium.

For  $0 < E_2 < k\alpha^2/4$ , the particle is free to move in the  $-x$  direction if  $x$  is less than point 3, in the  $+x$  direction if  $x$  is greater than point 8, and is bound if  $x$  lies between points 5 and 6. It is forbidden to exist between points 3 and 5 and 6 and 8.

For  $E_3 = 0$ , the particle is free to move in the  $-x$  direction if  $x < -\sqrt{2}\alpha$  (point 2), in the  $+x$  direction if  $x > \sqrt{2}\alpha$  (point 9), and is in stable equilibrium at the origin. It is forbidden to exist anywhere between  $\pm\sqrt{2}\alpha$  (points 2 and 9) except at the origin.

If  $E_4 < 0$ , the particle is forbidden between points 1 and 10, but is free to move in the  $-x$  direction if  $x$  is less than point 1 and in the  $+x$  direction if  $x$  is greater than point 10.

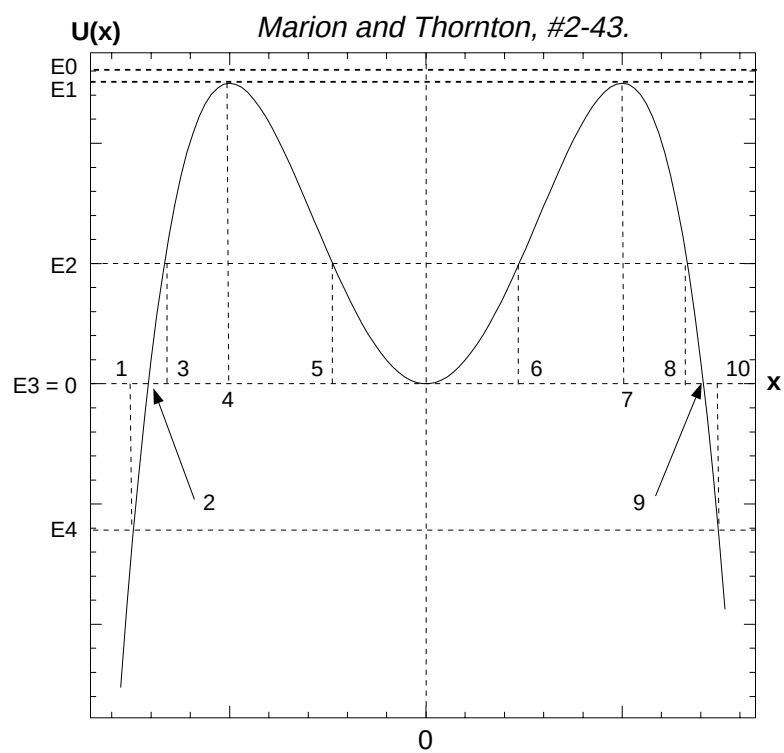


Figure 2: Plot for problem Marion and Thornton #2.43.