

# Homework 7

## Physics 2140

### Methods in Theoretical Physics

#### Due: Wednesday March 9, 2005

**Reading Assignment:** Grad, Div, Curl, Line Integrals: Boas, Ch. 6.6-6.8. Thornton and Marion, continue Ch. 2.

1. Boas 6.7.10.

Does  $\phi = \ln(x^2 + y^2)$  satisfy Laplace's equation ( $\nabla^2\phi = 0$ )?

$$\frac{\partial\phi}{\partial x} = \frac{2x}{x^2 + y^2} \qquad \frac{\partial^2\phi}{\partial x^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} \quad (1)$$

$$\frac{\partial\phi}{\partial y} = \frac{2y}{x^2 + y^2} \qquad \frac{\partial^2\phi}{\partial y^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} \quad (2)$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \quad (3)$$

Thus,  $\phi(x, y)$  satisfies Laplace's equation.

2. Boas 6.8.1.

Evaluate the line integral  $\int[(x^2 - y^2)dx - 2xydy]$  along each of the following paths from (0,0) to (1,2).

(i)  $y = 2x^2$ , (ii)  $x = t^2, y = 2t$ , (iii)  $y = 0$  from  $x = 0$  to  $x = 2$  then along the straight line joining (2,0) to (1,2).

(i) Along  $y = 2x^2$ , let  $x$  be the integration variable. The  $x$  limits are from 0 to 1. We substitute  $y = 2x^2, dy = 4xdx$  into the integral:

$$\int[(x^2 - y^2)dx - 2xydy] = \int_0^1 [(x^2 - 4x^4)dx - 2x \cdot 2x^2 \cdot 4xdx] \quad (4)$$

$$= \int_0^1 (x^2 - 20x^4)dx = -\frac{11}{3}. \quad (5)$$

(ii) Along the parametrically represented path  $x = t^2, y = 2t$ , let  $t$  be the integration variable. At (0,0),  $t = 0$  and at (1,2),  $t = 1$ . Along this path  $dx = 2tdt, dy = 2dt$ , so;

$$\int[(x^2 - y^2)dx - 2xydy] = \int_0^1 [(t^4 - 4t^2) \cdot 2tdt - 2t^2 \cdot 2t \cdot 2dt] \quad (6)$$

$$= \int_0^1 (2t^5 - 16t^3)dt = -\frac{11}{3}. \quad (7)$$

(iii) Along  $y = 0$  from  $x = 0$  to  $x = 2$ , we have  $dy = 0$  so:

$$I_1 = \int[(x^2 - y^2)dx - 2xydy] = \int_0^2 x^2dx = \frac{8}{3}. \quad (8)$$

Along the straight line from  $(2, 0)$  to  $(1, 2)$ , we have  $y = 4 - 2x$  (the equation of a straight line through the two points), so that:

$$I_2 = \int [(x^2 - y^2)dx - 2xydy] \quad (9)$$

$$= \int_2^1 [(x^2 - (4 - 2x)^2)dx - 2x(4 - 2x)(-2dx)] \quad (10)$$

$$= \int_2^1 (-11x^2 + 32x - 16)dx = -\frac{19}{3}. \quad (11)$$

So the entire integral along both paths is just the sum of the integrals along the individual paths:

$$\int [(x^2 - y^2)dx - 2xydy] = I_1 + I_2 = -\frac{11}{3}. \quad (12)$$

NOTE: A comparison of equations (4), (6), (12) reveals that the three path integrals have the same value. This is not an accident. It arises from the fact that we are integrating an exact differential since:

$$\frac{\partial}{\partial y} (x^2 - y^2) = -2y = \frac{\partial}{\partial x} (-2xy). \quad (13)$$

Another way to say this is that if we write the integral as  $\int dW = \int \mathbf{F} \cdot d\mathbf{r}$ , then this integral is path independent. Such a function  $\mathbf{F}$  is said to be conservative since, if  $\mathbf{F}$  is a force, then the work is independent of the path chosen. This condition is also equivalent to  $\nabla \times \mathbf{F} = 0$ . There is another powerful result that the next question explores, and that's that if a force  $\mathbf{F}$  is conservative then there exists a potential function  $\phi$  such that  $\mathbf{F} = -\nabla\phi$ . These facts will be discussed further as the class progresses, but it is important to know how to do curls to test if a specific force is conservative, how to find the potential function if the force is conservative (next question), and how to do line integrals to find the work along an arbitrary path.

### 3. Boas 6.8.9.

Verify that the force field  $\mathbf{F} = (3x^2yz - 3y)\mathbf{i} + (x^3z - 3x)\mathbf{j} + (x^3y + 2z)\mathbf{k}$  is conservative, then find a scalar potential  $\phi$  such that  $\mathbf{F} = -\nabla\phi$ .

A vector field  $\mathbf{F}$  is conservative if and only if  $\nabla \times \mathbf{F} = 0$ , and in fact for the function  $\mathbf{F}$  given here:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2yz - 3y & x^3z - 3x & x^3y + 2z \end{vmatrix} = 0, \quad (14)$$

so  $\mathbf{F}$  is conservative.

Now, we want to find a scalar potential function  $\phi(x, y, z)$  such that  $\mathbf{F} = -\nabla\phi$ ; that is,  $\phi = -\int \mathbf{F} \cdot d\mathbf{r}$ . If you recall, the value of a potential function has no meaning in and of itself, only its gradient is meaningful. In particular, an arbitrary constant can be added to a potential function everywhere without changing its gradient, so a potential function is said to have no absolute zero level. The potential function can be referenced to any point and set equal to any value at that point as long as the difference among values at various points remains constant. In our case now, let's choose the origin as our reference point and set the potential to zero at the origin. We then need to evaluate  $\phi = -\int \mathbf{F} \cdot d\mathbf{r}$  from the origin to the point  $(x, y, z)$ . We can do so along any path, since the function  $\mathbf{F}$  is conservative, so let's choose the path broken into the three line segments. Segment 1:  $d\mathbf{r}_1 = dx\hat{\mathbf{i}}$  with  $y = z = 0$ ; Segment 2:  $d\mathbf{r}_2 = dy\hat{\mathbf{j}}$  with  $z = 0$ ; Segment 3:  $d\mathbf{r}_3 = dz\hat{\mathbf{k}}$ .

The three segments transform the line integral into the sum of three integrals:

$$I_1 = - \int \mathbf{F} \cdot d\mathbf{r}_1 = - \int_0^x (3x^2 \cdot 0 \cdot 0 - 3 \cdot 0) dx = 0, \quad (15)$$

$$I_2 = - \int \mathbf{F} \cdot d\mathbf{r}_2 = - \int_0^y (x^3 \cdot 0 - 3x) dy = \int_0^y 3x dy = 3xy, \quad (16)$$

$$I_3 = - \int \mathbf{F} \cdot d\mathbf{r}_3 = - \int_0^z (x^3 y + 2z) dz = -x^3 y z - z^2. \quad (17)$$

Thus, the scalar potential is:

$$\phi = - \int \mathbf{F} \cdot d\mathbf{r} = I_1 + I_2 + I_3 = 3xy - x^3 y z - z^2. \quad (18)$$

4. Thornton and Marion 2-16.

Let the  $x$ -axis point up the slope. Then  $F_x = m\ddot{x} = m dv/dt = -mg \sin \theta$ , so

$$\int_{v_0}^v dv' = -g \sin \theta \int_0^t dt' \rightarrow v(t) = v_0 - (g \sin \theta)t. \quad (19)$$

When the object stops moving up the hill,  $v = 0$ , so  $t_{up} = v_0/g \sin \theta$ . The round trip, up then back, will be twice this value or

$$t = \frac{2v_0}{g \sin \theta} = 1.12 \text{ sec}, \quad (20)$$

where the numerical value results by inserting  $v_0 = 2.4 \text{ m/s}$  and  $\theta = 26^\circ$ .

5. Thornton and Marion 2-20. (Hint: you will need to use equations 2.54, 2.55 and Figure 2-3(c).)

First, note that we're given that  $m = 10 \text{ kg}$ ,  $v_0 = 140 \text{ m/s}$ , and  $R = 1000 \text{ m}$ . The equations we need are:

$$R = \frac{v_0^2}{g} \sin 2\theta \quad (21)$$

$$R' = R \left( 1 - \frac{4kv_0 \sin \theta}{3g} \right) \quad (22)$$

To find  $k$ , recall that drag  $F = mvk$ , so  $k = (F/vm)$ . We're told that  $m = 10 \text{ kg}$ . To find  $F/v$ , use Figure 2.3(c) at  $v = 140 \text{ m/s}$ . You see that the resistive force at this value of  $v$  is about  $30 \text{ N}$ , so  $k \sim 30/((140)(10) \text{ sec}^{-1}) \sim 0.02 \text{ sec}^{-1}$ . To solve for  $R'$ , we substitute equation (21) into (22). Doing so, however, results in a transcendental equation to find  $\theta$ . So, instead we'll aim for an approximate result by first estimating  $\theta$  in equation (21), by setting  $\theta = \theta_0$ , the value of the angle without air resistance. Without air resistance,  $\theta = 0.5 \sin^{-1} (gR/v_0^2) = 15^\circ$ , using the values given. Substituting this value for  $R$  in equation (22), solving for  $\theta$ , and inserting the numerical values given, we find that  $R' \sim 16.8^\circ$ .

6. Thornton and Marion 2-23. (Plotting the results with a computer as the question asks will give you extra credit.)

For numerical values, we're given that  $m = 1 \text{ kg}$ ,  $\alpha = 0.5 \text{ s}^{-1}$ , and  $k = 1 \text{ N/s}$ ,  $x_0 = v_0 = 0$ . We also told that  $F(t) = kte^{-\alpha t}$ . We find  $a(t)$ ,  $v(t)$ , and  $x(t)$  as follows:

$$a(t) = \frac{F(t)}{m} \rightarrow a(t) = \frac{k}{m} te^{-\alpha t} = \frac{dv}{dt} \quad (23)$$

$$v(t) = \int_0^v dv' = \int_0^t a(t') dt' = \frac{k}{m} \int_0^t t' e^{-\alpha t'} dt' = \frac{k}{m} \left[ -\frac{1}{\alpha^2} - \frac{1}{\alpha} \left( t + \frac{1}{\alpha} \right) e^{-\alpha t} \right] = \frac{dx}{dt} \quad (24)$$

$$x(t) = \int_0^x dx' = \int_0^t v(t') dt' = \frac{k}{m} \left[ -\frac{2}{\alpha^3} + \frac{t}{\alpha^2} + \frac{1}{\alpha^2} \left( t + \frac{2}{\alpha} \right) e^{-\alpha t} \right] \quad (25)$$

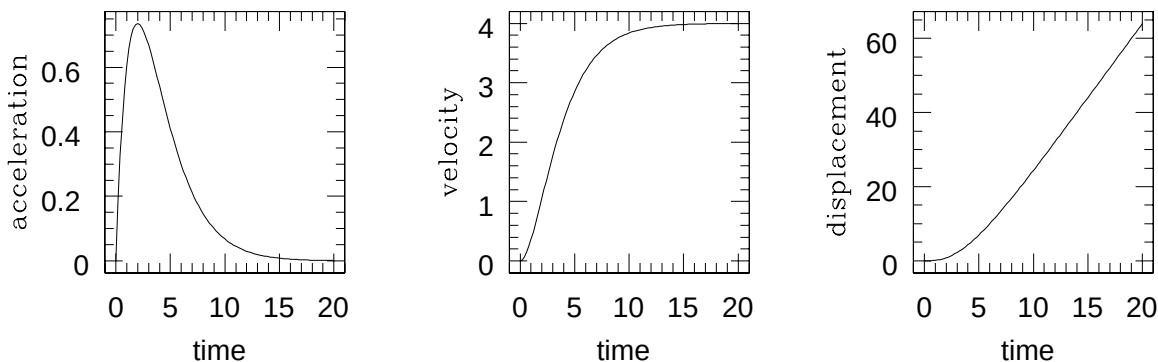
Substituting the values for  $m$ ,  $\alpha$ , and  $k$  given, we get:

$$a(t) = te^{-t/2} \quad (26)$$

$$v(t) = 4 - 2(t + 2)e^{-t/2} \quad (27)$$

$$x(t) = -16 + 4t + 4(t + 4)e^{-t/2} \quad (28)$$

Plotting these values we get the follow figures:



#### 7. Thornton and Marion 2-24.

We're given that  $m = 90$  kg, and  $\theta = 17^\circ$ . The problem can be done either with Newton's second law or conservation of energy. The latter we did in class, so we apply Newton's second law here. We need equations of motion for the skier on the hill and on the flat:

$$ma_h = F_g + F_{fr} = mg \sin \theta - \mu mg \cos \theta \rightarrow a_h = g(\sin \theta - \mu \cos \theta) \quad (\text{Hill}) \quad (29)$$

$$ma_f = F_{fr} = -\mu mg \rightarrow a_f = -\mu g \quad (\text{Flat}) \quad (30)$$

Let  $d = 100$  m be the distance that the skier skis down the hill and  $D = 70$  m be the distance from the base of the hill until he stops. Because he starts from rest, the skier's velocity at the base of the hill will be:

$$v_B^2(d) = 2a_h d = 2gd(\sin \theta - \mu \cos \theta) \quad (31)$$

Similarly, when the skier stops:

$$0 = v^2(D) = v_B^2 + 2a_f D = 2gd(\sin \theta - \mu \cos \theta) - 2\mu g D \quad (32)$$

Solving for  $\mu$  and substituting in the given values we get:

$$\mu = \frac{d \sin \theta}{d \cos \theta + D} = 0.18 \quad (33)$$

$$v_B^2 = 2a_f D = 2\mu g D \rightarrow v_B = \sqrt{2\mu g D} \rightarrow v_B = 15.6 \text{ m/s.} \quad (34)$$

#### 8. Thornton and Marion 2-27.

We're given that the rope's mass  $m = 0.4$  kg and its length  $L = 4$  m, so the density of the rope  $\rho = m/L = 0.1$  kg/m. Define a coordinate system whose origin is on the floor a distance  $L$  m from the table's surface. The rope initially hangs all the way to the floor. Consider a mass element  $dm(z)$  as a height  $z$ . The work needed to lift this mass element to the table is  $dW = dm(z)g(L - z) =$

$(\rho dz)g(L - z)$ . Now, we have to integrate over all mass elements extending from the floor ( $z = 0$ ) to the table ( $z = L$ ) to find the total work:

$$W = \int_0^W dW' = \rho g \int_0^L (L - z) dz = \frac{1}{2} \rho g L^2 = 0.18 \text{ J}, \quad (35)$$

where we have inserted  $L = 4 \text{ m}$ .