Homework 4
Physics 2140
Methods in Theoretical Physics
Due: Wednesday February 9, 2004

Reading Assignment from Boas:

Linear First Order ODEs.

1.

(a) Boas 8.3.6. The ODE can be rewritten as

$$y' + \frac{x}{\sqrt{x^2+1}} y = \frac{x}{\sqrt{x^2+1}}, \tag{1}$$

$$\tag{2}$$

so $Q(x) = P(x) = x/\sqrt{x^2+1}$. Thus,

$$\begin{aligned} I(x) &= \int P(x)dx = \int \frac{x}{\sqrt{x^2+1}} dx = \sqrt{1+x^2}, & (3)\\ y(x) &= e^{-I(x)} \left[\int Q(x) e^{I(x)} dx + c \right] = e^{-\sqrt{1+x^2}} \left[\int \frac{x}{\sqrt{x^2+1}} e^{\sqrt{1+x^2}} dx + c \right] & (4)\\ &= e^{-\sqrt{1+x^2}} \left(e^{\sqrt{1+x^2}} + c \right) = 1 + ce^{-\sqrt{1+x^2}}. & (5) \end{aligned}$$

(b) Boas 8.3.8. Divide the differential equation by $x \ln x$ to get it in the form of $y' + P(x)y = Q(x)$.

$$y' + \frac{1}{x \ln x} \, y = \frac{1}{x}. \tag{6}$$

$$\begin{aligned} P(x) &= \frac{1}{x \ln x}, \qquad Q(x) = \frac{1}{x}, & (7)\\ I(x) &= \int \frac{dx}{x \ln x} = \ln(\ln x), & (8)\\ e^{I(x)} &= e^{\ln(\ln x)} = \ln x, & (9)\\ y(x) e^{I(x)} &= \int e^{I(x)} Q(x) dx + C \to y(x) \ln x = \int \frac{\ln x}{x} \, dx + C = \frac{(\ln x)^2}{2} + C, & (10)\\ y(x) &= \frac{1}{2} \ln x + \frac{C}{\ln x}. & (11) \end{aligned}$$

Taylor and MacLaurin Series.

2. Boas 1.13.13.

$$
\begin{array}{ll}
f(x) = \sin x^2 & f(0) = 0 \\
f'(x) = 2x\cos x^2 & f'(0) = 0 \\
f''(x) = 2\cos x^2 - 4x^2 \sin x^2 & f''(0) = 2 \\
f'''(x) = -12x \sin x^2 - 8x^3 \cos x^2 & f'''(0) = 0 \\
f^{iv}(x) = -48x^2 \cos x^2 + 16x^4 \sin x^2 - 12\sin x^2 & f^{iv}(0) = 0 \\
f^{v}(x) = -120x\cos x^2 + 160x^3 \sin x^2 + 32x^5 \cos x^2 & f^{v}(0) = 0 \\
f^{vi}(x) = -120\cos x^2 + 720x^2 \sin x^2 + 480x^4 \cos x^2 - 64x^6 \sin x^2 & f^{vi}(0) = -120 \\
\vdots & \vdots
\end{array}
$$

$$
\begin{aligned}
\sin x^2 &= f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + \dots && (12)\\
&= 0 + 0 + 2\frac{x^2}{2!} + 0 + 0 + 0 - 120\frac{x^6}{6!} + \dots && (13)\\
&= x^2 - \frac{x^6}{6} + \dots && (14)
\end{aligned}
$$

To get the higher terms one should use series 13.1 given on page 24 of Boas remembering to replace x with x^2.

Application of Power Series.

3. Boas 1.15.10.

To evaluate $\int_0^1 (e^x - 1)/x dx$, use the series expansion for e^x then subtract 1 and divide by x. This is a lot easier than finding the Taylor Series expansion of the whole integrand:

$$
\begin{aligned}
\frac{1}{x}(e^x - 1) &= \frac{1}{x}\left(\sum_{n=0}^{\infty}\frac{1}{n!}x^n - 1\right) = \frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{1}{n!}x^n\right) = \sum_{n=1}^{\infty}\frac{1}{n!}x^{n-1} && (15)\\
\int_0^1 \frac{e^x - 1}{x}dx &= \sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1 x^{n-1}dx = \sum_{n=1}^{\infty}\frac{1}{n(n!)}x^n\bigg|_0^1 = \sum_{n=1}^{\infty}\frac{1}{n(n!)} \sim 1.31790 && (16)
\end{aligned}
$$

Because $1/n!$ goes so rapidly to 0, the series converges to about 1.31790 very fast.

4. Boas 1.15.28.

We're given:

$$
E = \frac{mc^2}{(1 - \frac{v^2}{c^2})^{1/2}} = mc^2(1 - \frac{v^2}{c^2})^{-1/2}. \quad (17)
$$

We want to find the Maclaurin Series expansion of $(1 - \frac{v^2}{c^2})^{-1/2}$ which is the same as finding the series expansion of $(1-x)^{-1/2}$ (binomial expansion) with $x = v^2/c^2$. This is just equation (13.5) on p.29 in Boas. We get

$$
(1 - \frac{v^2}{c^2})^{-1/2} = 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \frac{5v^6}{16c^6} + \frac{35v^8}{128c^8} + \dots \quad (18)
$$

Retaining terms only through second order in v/c gives:

$$E \sim mc^2 + \frac{1}{2}mv^2, \tag{19}$$

where the first term is called "rest-mass energy" and the second term is "kinetic energy". This says that if velocity is sufficiently small, the energy of an object is approximately its rest-mass energy and its kinetic energy.

5. **The Bouncing Ball.** Do problems *a* through *e* from section 2.3 found on p.13-15 of Snieder, *A Guided Tour of Mathematical Physics for the Physical Sciences.* The relevant section from Snieder's book was handed out with the homework assignment in class. p.9 of Snieder, which lists several Taylor series for well known functions, was also appended at the end of the homework.

(a) If the position variable z is positive upward, the acceleration of gravity is $a = -g$, and v_0 is the initial velocity when the ball is thrown from floor level ($z = 0$), then

$$\begin{aligned} a &= -g &(20)\\ v(t) &= v_0 - gt &(21)\\ z(t) &= v_0 t - \frac{1}{2}gt^2 &(22)\\ v^2(z) &= v_0^2 - 2gz. &(23)\end{aligned}$$

The derivation of these equations proceeds as in class by integrating with the appropriate initial conditions starting from equation (20).

The maximum height that the ball will reach, H, occurs when the ball has zero velocity, which from equation (23) occurs when $0 = v_0^2 - 2gH$, so

$$H = \frac{v_0^2}{2g}. \tag{24}$$

Also, the time t to return to the ground can be found from equation (22) by setting $z = 0$, so that $0 = v_0 - gt/2$ and:

$$t = \frac{2v_0}{g}. \tag{25}$$

Note that from equation (24), $v_0 = \sqrt{2gH}$, which we can use with (25) to rephrase t in terms of H rather than v_0:

$$t = \frac{2v_0}{g} = \frac{2\sqrt{2gH}}{g} = \sqrt{8H/g}. \tag{26}$$

(b) The kinetic energy of the n-th bounce is equal to $(1-\gamma)$ times the kinetic energy of the $(n-1)$-th bounce, where γ is the fraction of energy lost on each bounce:

$$\begin{aligned} \mathrm{KE}^{(n)} &= \mathrm{KE}^{(n-1)}(1-\gamma) &(27)\\ \frac{1}{2}mv_n^2 &= \frac{1}{2}mv_{n-1}^2(1-\gamma) &(28)\\ v_n &= \sqrt{1-\gamma}\,v_{n-1}. &(29)\end{aligned}$$

(c) From equation (25), the time for the n-th and the $(n-1)$-th bounces follows immediately, $t_n = 2v_n/g$ and $t_{n-1} = 2v_{n-1}/g$, so

$$v_{n-1} = \frac{gt_{n-1}}{2}. \tag{30}$$

So using equations (29) and (30),

$$
\begin{aligned}
t_n &= \frac{2v_n}{g} = \frac{2\sqrt{1-\gamma}v_{n-1}}{g} = \frac{2\sqrt{1-\gamma}gt_{n-1}}{2g} && (31)\\
&= 2\sqrt{1-\gamma}t_{n-1}. && (32)
\end{aligned}
$$

A pattern emerges when we consider equations (26) and (32) for a set of values of bounce number n:

$$
\begin{aligned}
t_0 &= \sqrt{8H/g} && (33)\\
t_1 &= \sqrt{1-\gamma}\sqrt{8H/g} && (34)\\
t_2 &= (1-\gamma)\sqrt{8H/g} && (35)\\
t_3 &= (1-\gamma)^{3/2}\sqrt{8H/g} && (36)\\
\vdots &= \vdots && (37)\\
t_n &= (1-\gamma)^{n/2}\sqrt{8H/g} && (38)
\end{aligned}
$$

(d) Using equation (38), the total time through the N-th bounce is:

$$T_N = \sum_{n=0}^{N} t_n = \sqrt{8H/g}\sum_{n=0}^{N}(1-\gamma)^{n/2} \tag{39}$$

Now recall the geometric series, $1/(1-x) = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$. If we let $x = \sqrt{1-\gamma}$, then

$$\frac{1}{1-\sqrt{1-\gamma}} = \sum_{n=0}^{\infty}(\sqrt{1-\gamma})^n = \sum_{n=0}^{\infty}(1-\gamma)^{n/2}. \tag{40}$$

Therefore, using equations (39) and (40), the total time for all the bounces is:

$$
\begin{aligned}
T_\infty &= \lim_{N\to\infty} T_N = \sum_{n=0}^{\infty} t_n = \sqrt{\frac{8H}{g}}\sum_{n=0}^{\infty}(1-\gamma)^{n/2} && (41)\\
&= \sqrt{\frac{8H}{g}}\frac{1}{1-\sqrt{1-\gamma}}. && (42)
\end{aligned}
$$

(e) T_∞ is finite, as equation (42) shows, unless $\gamma = 0$; that is, if there are no energy losses on each bounce. If $\gamma = 0$, then $T_\infty = \infty$.

6. **Distance to the Horizon.** Consider Figure 1. The purpose of this problem is to find an approximate formula for the distance s to the horizon along a sphere of radius R, given a height h above the surface. We wish to show that $s \sim \sqrt{2Rh}$.

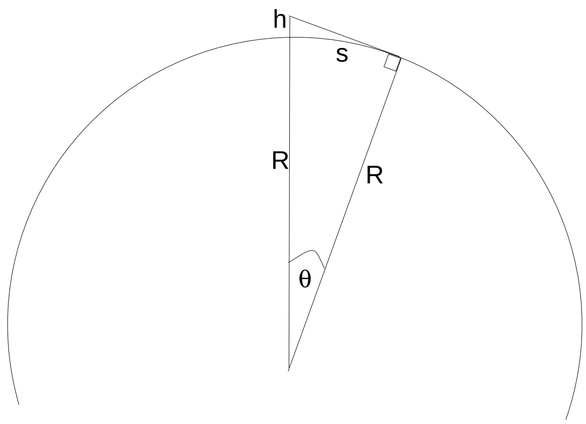

Figure 1:

(a) Show that $h/R = \sec\theta - 1$. (Hint: Write $\cos\theta$ in terms of h and R and then manipulate.)

$$\cos\theta = \frac{R}{R+h} \rightarrow \sec\theta = \frac{R+h}{R} = 1 + \frac{h}{R} \tag{43}$$

$$\frac{h}{R} = \sec\theta - 1 \tag{44}$$

(b) Show that for small θ $\sec\theta$ can be approximated by the following formula:

$$\sec\theta \sim 1 + \theta^2/2$$

(Hint: Find the first two non-zero terms in the Taylor series expansion (about $\theta = 0$) for $\sec\theta$.)

$f(\theta) = \sec\theta$	$f(0) = 1$
$f'(\theta) = \sec\theta\tan\theta$	$f'(0) = 0$
$f''(\theta) = \sec^3\theta + \sec\theta\tan^2\theta$	$f''(0) = 1$

Thus, $\sec\theta \sim f(0) + f'(0)\theta + f''(0)\theta^2/2 = 1 + \theta^2/2$.

(c) From your results in (a) and (b), show that $h/R \sim \theta^2/2$ for small θ.

$$\frac{h}{R} = \sec\theta - 1 \sim \left(1 + \frac{\theta^2}{2}\right) - 1 = \frac{\theta^2}{2} \tag{45}$$

(d) From your result in (c), show that $s \sim \sqrt{2hR}$. (Hint: recall the arc-length of a circle is $s = R\theta$.)

$$\theta \sim \sqrt{\frac{2h}{R}} \rightarrow s \sim R\theta = \sqrt{2hR} \tag{46}$$

(e) Apply the result to Green Mountain. For simplicity, assume that $h = 1$ km and find the distance to the horizon s. (For Earth, $R \sim 6370$ km.)

$s \sim \sqrt{2hR} = \sqrt{2(6371)}$ km ~ 110 km.