

Homework 12
Physics 2140
Methods in Theoretical Physics
Due: Friday April 29, 2005

1. Consider a traveling wave on a string moving in the $+x$ -direction as follows: $y(x, t) = \sin(kx - \omega t)$. Show that $y(x, t)$ is a solution to the 1-D wave equation for a string; where the 1-D wave equation is:

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y(x, t)}{\partial t^2}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= -k^2 \sin(kx - \omega t) = -k^2 y \\ \frac{\partial^2 y}{\partial t^2} &= -\omega^2 \sin(kx - \omega t) = -\omega^2 y \end{aligned}$$

(1)

from which the desired result follows immediately because $c = \omega/k$.

2. **Application of Initial Conditions to the String Equation.** Suppose a string of length L is initially displaced a small distance b at the middle of the string and then released (i.e., it's plucked in the middle). Determine the subsequent motion of the string. (Hint: The initial conditions translate to $y(x, 0) = 2bx/L$ on $0 \leq x \leq L/2$, $y(x, 0) = 2b(L - x)/L$ on $L/2 < x \leq L$, and $\dot{y}(x, 0) = 0$ on $0 \leq x \leq L$.)

We start with the general solution to the string equation with ends clamped at $x = 0$ and $x = L$:

$$y(x, t) = \sum_{n=1}^{\infty} \sin(k_n x) (A_n \cos \omega_n t + B_n \sin \omega_n t) \quad (2)$$

where $k_n = n\pi/L$ and $\omega_n = ck_n = n\pi c/L$. From the initial velocity, we know that $B_n = 0$. We need only expand the initial displacement $y(x, 0) = f(x)$ in a Fourier Series $f(x) = \sum_{n=1}^{\infty} a_n \sin k_n x$ where:

$$A_n = a_n = \frac{2}{L} \left[\frac{2b}{L} \int_0^{L/2} x \sin(n\pi x/L) dx + \frac{2b}{L} \int_{L/2}^L (L - x) \sin(n\pi x/L) dx \right] = \frac{8b}{n^2 \pi^2} \sin\left(\frac{n\pi}{2}\right) \quad (3)$$

where we used the fact that $\int x \sin(ax) dx = (x^2/2) \sin ax$ to establish the value of the integral. Now, $\sin(n\pi/2)$ is zero when n is even and is ± 1 when n is odd. In particular, it is equal to $+1$ when $n = 1, 5, 9, 13, \dots$ and equal to -1 when $n = 3, 7, 11, \dots$. If we let n range from 0 to ∞ , and replace it with $(2n + 1)$, we can set $\sin(n\pi/2) = (-1)^{2n+1}$. Thus,

$$A_n = (-1)^{2n+1} \frac{8b}{(2n + 1)^2 \pi^2} \quad (4)$$

The particular solution, therefore, is:

$$y(x, t) = \frac{8b}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{(2n + 1)^2} \sin\left(\frac{(2n + 1)\pi x}{L}\right) \cos\left(\frac{(2n + 1)\pi ct}{L}\right). \quad (5)$$

3. **Separation of variables applied to the Schrodinger equation.** Given the time-dependent Schrodinger equation for the wavefunction $\Psi(x, t)$:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U(x)\Psi(x, t) = i\hbar \frac{\partial \Psi}{\partial t}. \quad (6)$$

Let

$$\Psi(x, t) = \psi(x)T(t). \quad (7)$$

Using separation of variables, derive the time-independent Schrodinger equation for the spatial wavefunction $\psi(x)$:

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \quad (8)$$

Remember to let E be the separation constant.

To substitute equation (7) into (6) we'll need the following derivatives:

$$\begin{aligned} \frac{\partial^2 \Psi(x, t)}{\partial x^2} &= T(t) \frac{\partial^2 \psi(x)}{\partial x^2} = T(t) \frac{d^2 \psi(x)}{dx^2} \\ \frac{\partial \Psi(x, t)}{\partial t} &= \psi(x) \frac{\partial T(t)}{\partial t} = \psi(x) \frac{dT(t)}{dt} \end{aligned}$$

The partial derivatives are replaced with ordinary derivatives at the end of each equation because the function that's being differentiated is a function only of a single variable. So, we substitute these equations into equation (6):

$$-\frac{\hbar^2}{2m} T(t) \frac{d^2 \psi(x)}{dx^2} + U(x)\psi(x)T(t) = i\hbar \psi(x) \frac{dT(t)}{dt}$$

Next we rearrange to separate the functions of t and x by dividing both sides of the equation by $\psi(x)T(t)$:

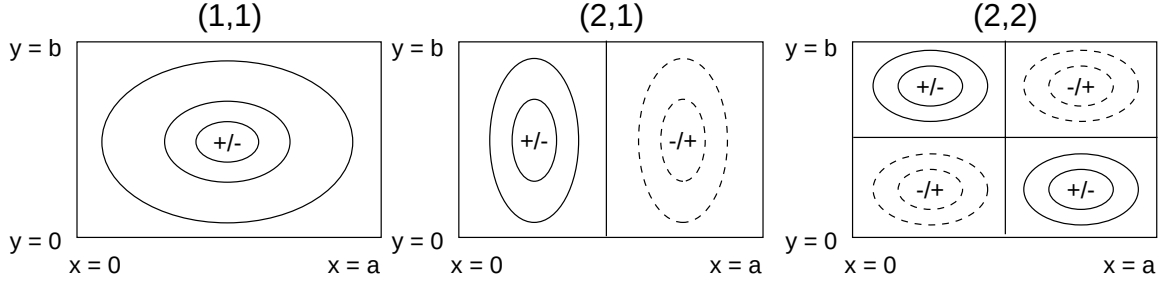
$$-\frac{\hbar^2}{2m} \frac{1}{\psi(x)} \frac{d^2 \psi(x)}{dx^2} + U(x) = i\hbar \frac{1}{T(t)} \frac{dT(t)}{dt} \quad (9)$$

Note that in equation (9), we have a function of x equal to a function of t . The only way this can be true in general is if both sides of the equation are equal to the same constant. Because both sides of the equation have the units of energy, let's let the constant that both sides of equation (9) equals be called E . If we do that, equation (6) divides into two equations:

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{1}{\psi(x)} \frac{d^2 \psi(x)}{dx^2} + U(x) &= E \\ i\hbar \frac{1}{T(t)} \frac{dT(t)}{dt} &= E \end{aligned}$$

Multiplying the first equation by $\psi(x)$, we get the desired result, equation (8).

4. In the notes titled *The Wave Equation in 2-D*, the general solution for the 2-D wave equation in Cartesian coordinates with clamped boundaries is given by equation (30).
- (a) If (n, m) describes a mode of oscillation, draw the following modes $(1, 1)$, $(2, 1)$, and $(2, 2)$.
 - (b) What are the frequencies of the three modes in (a)?



From eqn (31) in your notes,

$$\omega_{nm} = c\pi \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right)^{1/2}. \quad (10)$$

Thus, the frequencies are:

$$\omega_{11} = \frac{c\pi}{ab} (a^2 + b^2)^{1/2} \quad (11)$$

$$\omega_{21} = \frac{c\pi}{ab} (a^2 + 4b^2)^{1/2} \quad (12)$$

$$\omega_{22} = \frac{2c\pi}{ab} (a^2 + b^2)^{1/2} = 2\omega_{11} \quad (13)$$

(c) Under what condition will the frequencies of the mode (1, 2) be equal to the frequencies of the mode (2, 1)?

$\omega_{12} = \omega_{21}$ if $a = b$; i.e., the membrane is a square.

(d) The equality of frequencies in question (c) is an example of a *degeneracy*. Degeneracies are typically related to a spatial *symmetry* exhibited by the system. What is the symmetry exhibited by the system if the frequencies of the mode (1, 2) are equal to the frequencies of the mode (2, 1)?

If the membrane is a square, then we have 90° rotational symmetry.

(e) Complete the derivation by filling in the missing steps to find the coefficients a_{nm} in equation (34). (The derivation of the coefficients b_{nm} seen in equation (37) is similar, so you don't have to present it.)

From eqn (33) in your notes,

$$f(x, y) = \sum_n \sum_m a_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right). \quad (14)$$

Now we take advantage of the orthogonality of sinusoids and multiply both sides by

$$\sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{m'\pi y}{b}\right),$$

and then integrate over the area of the membrane. Noting that $\int_0^b \sin\left(\frac{n\pi x}{b}\right) \sin\left(\frac{m\pi x}{b}\right) dx = \frac{b}{2} \delta_{mn}$, we get

$$\int_0^b \int_0^a f(x, y) \sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{m'\pi y}{b}\right) dx dy \quad (15)$$

$$\begin{aligned}
&= \sum_n \sum_m a_{nm} \int_0^b \int_0^a \left[\sin\left(\frac{m'\pi y}{b}\right) \sin\left(\frac{m\pi y}{b}\right) \right] \left[\sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \right] dx dy \\
&= \sum_n \sum_m a_{nm} \left[\frac{b}{2} \delta_{mm'} \right] \left[\frac{a}{2} \delta_{nn'} \right] \tag{16}
\end{aligned}$$

$$= \frac{ab}{4} a_{n'm'}.$$
 \tag{17}

Changing the n' and m' to n and m in equations (12) and (14), and then multiplying both sides by $4/ab$, we get the desired result.

5. Consider the vibrations of a rectangular membrane as described in section 1 of the notes *The Wave Equation in 2-D*.

(a) If the membrane is initially displaced from rest according to:

$$u(x, y, 0) = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right),$$

what will be the particular solution to the problem?

The solution is given by eqn (35) in the notes. We solve for the coefficients by putting the given initial condition into $f(x, y)$ in eqn (37) in the notes. This gives us:

$$a_{nm} = \frac{4}{ba} \int_0^a \int_0^b f(x, y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dy dx \tag{18}$$

$$= \frac{4}{ba} \int_0^a \int_0^b \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) dy dx. \tag{19}$$

And using the same orthogonality conditions we used in problem (2.e), we find that $a_{nm} = \delta_{n,1} \delta_{m,1}$. Therefore,

$$a_{nm} = \begin{cases} 1 & \text{if } m = n = 1 \\ 0 & \text{if } m \neq 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

Also, note that $b_{nm} = 0$ because $\dot{u}(x, y, 0) = 0$. So by eqn (30) in the notes, the solution is

$$u(x, y, t) = \cos(\omega_{11}t) \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right), \tag{20}$$

where $\omega_{11} = \frac{c\pi}{ab} (a^2 + b^2)^{1/2}$

(b) Explain physically why there is only one term in the solution.

Because the initial displacement is only in the (1, 1) mode, then it is the only mode which it excites.