

Homework 10
Physics 2140
Methods in Theoretical Physics
Due: Wednesday April 13, 2005

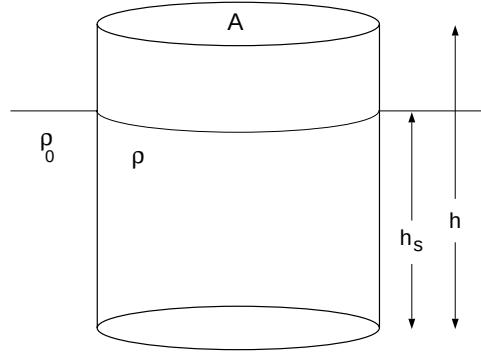
1. Thornton and Marion.

(a) 3-1. $m = 100$ g, $k = 10^4$ dyne, $x = 3$ cm, $v_0 = 0$. $\tau_0 = 2\pi/\omega_0 = 2\pi\sqrt{m/k} \sim 0.63$ sec. $E = (1/2)kx^2 = 4.5 \times 10^4$ erg. $E = (1/2)mv_{\max}^2 \rightarrow v_{\max} = \sqrt{2E/m} = 30$ cm/sec.

(b) 3-2. $T = 10$ sec, $A(t+T)/A(t) = 1/2$. $\exp(-\beta T) = 1/2 \rightarrow \beta = -\ln(1/2)/T = \ln(2)/T = 6.9 \times 10^{-2}$ rad/sec. $\omega_1 = \sqrt{\omega_0^2 - \beta^2} = [10^2 - (6.9 \times 10^{-2})^2]^{1/2} = 10 [(1 - (6.9)^2 10^{-6})]^{1/2} \sim 10(1 - (1/2)(6.9)^2 10^{-6})$ so $\nu_1 \sim (10/2\pi)(1 - (1/2)(6.9)^2 10^{-6})$. But $\nu_0 = 10/(2\pi)$ Hz, so the shift in the frequency caused by adding damping is $\nu_0 - \nu_1 \sim (1/2)(6.9)^2 10^{-6} \sim 2.4 \times 10^{-5}$ Hz. The decrement of motion is $\exp(\beta/\nu_1) \sim 1.045$.

(c) 3-3. $\nu_0 = 1$ cm/s, $v_0 = 1$ cm/s, $x_0 = 0$. $E_{\text{tot}} = (1/2)mv_0^2 = (1/2)kx_{\max}^2 \rightarrow x_{\max} = v_0\sqrt{m/k} = 0.1$ cm. $U_{\max} = (1/2)kx_{\max}^2 = 50$ erg.

2. Thornton and Marion. 3-7.



This is the oscillating iceberg problem. There are two parts. The first is to write down the equation of equilibrium that will give us a relation between the density of the berg ρ , the density of the water ρ_0 , the amount of the berg that's submerged h_s , and the total height of the berg h . The second part is to write down the dynamical equation in which the ice berg is displaced by a small height x , so that $h_s \rightarrow h_s + x$. In this case, the berg will not be in equilibrium but will experience a net restoring force and will oscillate about its equilibrium position.

First, in equilibrium the ice berg experiences no net force. Archimedes Principle states that its weight will be perfectly offset by the buoyancy force of water which is just the weight of water that is displaced by the ice berg:

$$\begin{aligned} F_{\text{net}} = 0 &= F_{\text{grav}} + F_{\text{buoy}} = (\text{Mass of iceberg})g + (\text{Mass of water displaced})g \\ &= (\rho Ah)g - (\rho_0 Ah_s)g \rightarrow h_s = h \left(\frac{\rho}{\rho_0} \right) \end{aligned} \quad (1)$$

Second, now let $h_s \rightarrow h_s + x$:

$$\begin{aligned} F_{\text{net}} = M\ddot{x} &= \rho Ahg - \rho_0 A(h_s + x)g = \rho Ahg - \rho_0 A(h(\rho/\rho_0) + x)g \\ &= \rho Ahg - \rho Ahg - \rho_0 A xg = -(\rho_0 Ag)x \end{aligned} \quad (2)$$

where we used equation (1) along the way. Therefore, from equation (1) we see that:

$$\ddot{x} + \left(\frac{\rho_0 A g}{M} \right) x = 0.$$

This is just the simple harmonic oscillator equation where

$$\omega_0 = \sqrt{\frac{\rho_0 A g}{M}}$$

To get this in the form required, we note that the mass of the berg:

$$M = \rho V = \rho A h \stackrel{(1)}{=} \rho A h_s (\rho_0 / \rho) = \rho_0 h_s A = \rho_0 V_s$$

where V is the total volume of the berg and V_s is the volume that's submerged. Substituting this expression for the mass into the frequency equation:

$$\omega_0 = \sqrt{\frac{g A}{V_s}}$$

If, as given, $A = 1 \text{ cm}^2$, $\rho = 0.8 \text{ g/cm}^3$, and $\rho_0 = 1 \text{ g/cm}^3$, and $V_s = 0.8 \text{ cm}^3$, then $\tau = 2\pi/\omega_0 \sim 0.18 \text{ sec}$.

3. Thornton and Marion. 3-10.

$$\begin{aligned} e^{-1} &= -\beta n \tau_1 \rightarrow 1 = \beta n \tau_1 \rightarrow \beta = \frac{1}{n \tau_1} = \frac{\omega_1}{2\pi n} \\ \omega_1^2 &= \omega_0^2 - \beta^2 = \omega_0^2 - \left(\frac{1}{2\pi n} \right)^2 \omega_1^2 \\ \omega_1^2 &= \left(1 + \frac{1}{4\pi^2 n^2} \right) \omega_0^2 \\ \omega_1 &= \left(1 + \frac{1}{4\pi^2 n^2} \right)^{-1/2} \omega_0 \\ \omega_1 &\sim \left(1 - \frac{1}{8\pi^2 n^2} \right) \omega_0 \end{aligned}$$

4. Thornton and Marion. 3-14. (Hint: you know that $\cosh y = (e^y + e^{-y})/2$ and $\sinh y = (e^y - e^{-y})/2$, so $e^y = \cosh y + \sinh y$ and $e^{-y} = \cosh y - \sinh y$.)

$$\begin{aligned} x(t) &= e^{-\beta t} [A e^{\omega_2 t} + B e^{-\omega_2 t}] \\ &= e^{-\beta t} [A (\cosh \omega_2 t + \sinh \omega_2 t) B (\cosh \omega_2 t - \sinh \omega_2 t)] \\ &= e^{-\beta t} [C \cosh \omega_2 t + D \sinh \omega_2 t] \end{aligned} \tag{3}$$

where $C = A + B$ and $D = A - B$. In addition, the $e^{-\beta t}$ can be expanded into the difference between a cosh and sinh and multiplied through.

To find $\dot{x}(t)$, recall that $d/dt(\cosh \omega t) = \omega \sinh \omega t$ and $d/dt(\sinh \omega t) = \omega \cosh \omega t$. Then from equation (3):

$$\begin{aligned}\dot{x}(t) &= -\beta e^{-\beta t} [C \sinh \omega_2 t + D \cosh \omega_2 t] + \omega_2 e^{-\beta t} [C \sinh \omega_2 t + D \cosh \omega_2 t] \\ &= e^{-\beta t} [(-\beta C + \omega_2 D) \cosh \omega_2 t + (-\beta D + \omega_2 C) \sinh \omega_2 t]\end{aligned}$$

where again the $e^{-\beta t}$ can be expanded into the difference between a cosh and sinh and multiplied through so that the entire expression can be given in terms of sinh and cosh.

5. Find the particular solution to the following forced, underdamped ($\omega_0 > \beta$) equations. Note that because the right-hand-side of each equation is real, the particular solutions will be real.

(a) $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F_0/m$ where F_0 is a real constant.

Try $x_p(t) = A$. Substituting into the ODE and solving for A we get $x_p(t) = A = F_0/(\omega_0 m)$.

(b) $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{-\alpha t}$, where α is a real positive number and F_0 is a real constant.

Try $x_p(t) = A \exp(-\alpha t)$. Substituting into the ODE and solving for A we get $A = (F_0/m) (\alpha^2 - 2\beta\alpha + \omega_0^2)^{-1}$, so $x_p(t) = (F_0/m) (\alpha^2 - 2\beta\alpha + \omega_0^2)^{-1} \exp(-\alpha t)$.