Space-Time Diagrams, Relativity of Simultaneity, and Relative Time Reversal

Recall the Lorentz transformations between two frames in which frame S' is moving in the position x-direction relative to S with speed v:

$$\begin{aligned} x' &= \gamma\,(x - vt) \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}$$

These transformations show that space and time are inextricably interwoven in special relativity.

The implications of the Lorentz transformations can sometimes be made clearer with a space-time diagram, which in two of the four dimesions is shown in Figure 1. Note how the time axis is converted to distance by multiply by c. This diagram is for a particular frame S and the space-time diagrams for all other frames will be different. We will not consider how space-time diagrams transform for different frames, which is somewhat complicated and beyond the scope of the current class.

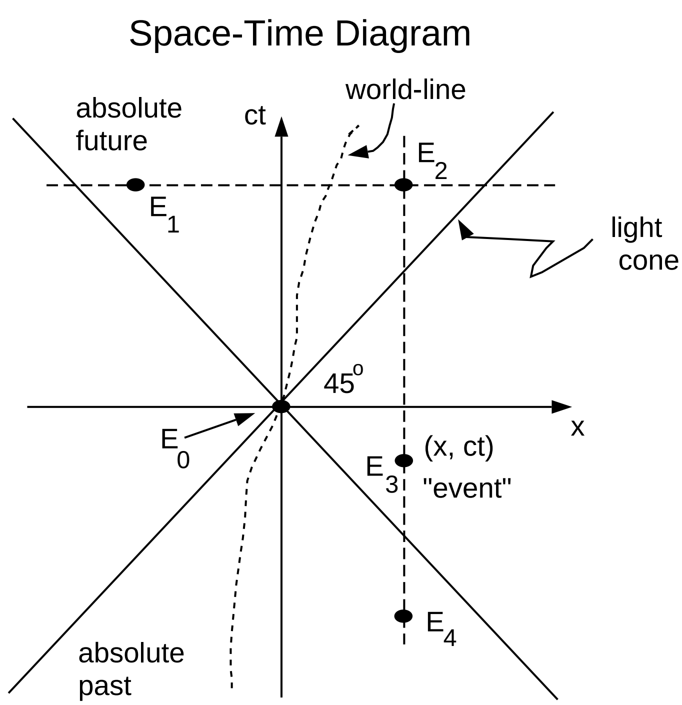

Figure 1: Space-time diagram

Each point in the diagram is a space-time *event*. The origin is usually considered to be the "current" event for the diagram which we denote E_0. Light beams through the origin trace lines at 45° to the coordinate axes because $x = ct$. These lines define the *light cone*. Any massive body that passes through the origin, will trace a *world-line* inside the light cone.

Events in the diagram fall into three main categories: those inside ($x < ct$), those on ($x = ct$), and those outside of the light cone ($x > ct$). Physical signals must travel no faster than the speed of light and, therefore, communication with event E_0 can only take place with events inside or on the light cone. For this reason, the inside of the light cone is called the *absolute future* and *absolute past* for event E_0 because only physical processes at events in this region can communicate or interact with processes at E_0. Therefore, nothing could happen at E_3 to affect a process at E_0. A process at E_3 could affect things at E_2, but E_2 is in the abolute future of E_0 so it cannot affect things at E_0 although things at E_0 could affect processes at E_2.

Similarly, space-time intervals between events are of three types: *space-like* intervals like that between E_1 and E_2 for simultaneous events, *time-like* intervals like that between E_2, E_3, and E_4 for events at the same spatial location, and *space-time* intervals, which is every other interval. Note that because of the relativity of simultaneaity, events that are time-like in one frame will generatlly not bein other frames. Similarly, events observed to be in the same location in S will generally be in other frames. That is, time-like or space-like intervals in one frame may not be in other frames.

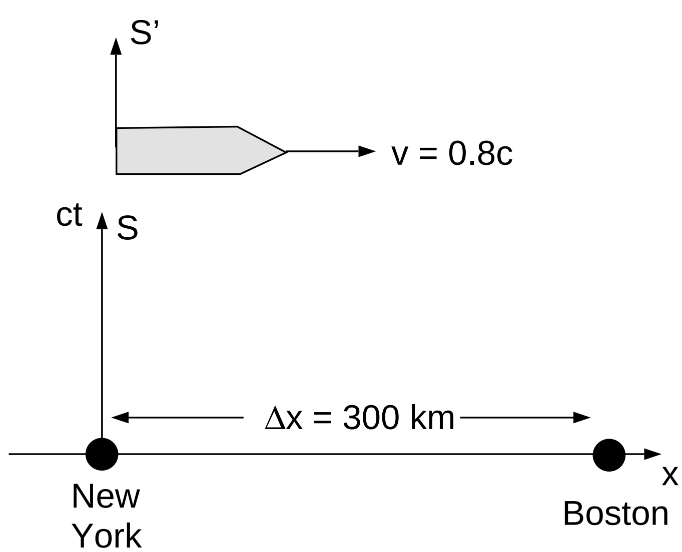

Figure 2: Example of relativity of simultaneity.

As an example, consider the situation in Figure 2 in which we compare the time between two events, one in New York and one in Boston. These events are observed in the frame of the earth to be separated by a distance $\Delta x = 300$ km and to occur simultaneously; i.e., $\Delta t = 0$. To be more specific, let's assume that the events are the lights going on in Boston and New York. The time separation observed in the rocket's frame is $\Delta t' = \gamma(\Delta t - v\Delta x/c^2) = -\gamma v \Delta x/c^2 = (5/3)(-.8(300)/(3\times10^5)) \sim -1.3$ ms. That is, the lights are observed to go on in New York about 1.3 ms later than in Boston. This shows that "simultaneity" is reference frame dependent; i.e., simultaneity is relative.

Going further, let's assume instead that in the earth's frame the lights are not observed to go on simultaneously in Boston and New York, but actually go on 1 ms later in Boston than in New York. That is $\Delta t = t_B - t_{NY} = 1$ ms. In the rockets' frame, $\Delta t' = \gamma(\Delta t - v\Delta x/c^2) = -(5/3)(10^{-3} - (-.8(300)/(3 \times 10^5)) = (1$ - 1.3) ms = - 0.3 ms. That is, observers in the rocket observe the lights to come on later in New York than Boston; the reverse order of what observers in the Earth's frame see. Thus, not only is simultaneity relative, time ordering can be relative.

This "relativity of time ordering" is potentially more troubling than the relativity of simultaneity, although they're statements of the same phenomenon, because apparently cause and effect can be observed to change order. How can the laws of physics be invariant relative to changes in reference frame if cause and effect are not well ordered?

The answer to this question lies in an inpsection of the equation:

$$\Delta t' = \gamma(\Delta t - v\Delta x/c^2).$$

Assuming that Δt is positive, $\Delta t'$ can have the opposite sign from Δt only if $\Delta t - v\Delta x/c^2 < 0$; i.e., if

$$\Delta t < v\Delta x/c^2. \tag{1}$$

Let's investigate equation (1) by considering the relationship between the two events separated by Δx and Δt in the earth's frame in terms of the speed of a signal needed to communicate between the two events: $u = \Delta x/\Delta t = \beta c$. In these terms then, the condition for relative time reversal, equation (1), can be rewritten as:

$$\beta \frac{v}{c} > 1.$$

Because $v/c < 1$ always, $\beta > 1$. That is, relative time reversal can only occur if Δx is greater than the distance light can travel in time Δt. In terms of Figure 1, relative time reversal could occur between E_0 and E_3, but not between E_0 and any other event. In fact, $\beta > c$ is a necessary but not a sufficient condition for relative time reversal; $\beta > c/v$. Even if one could send signals faster than c, one would not be guaranteed in reversing relative time ordering.

Therefore, relative time reversal can only occur between events that cannot communicate by any physically realistic signal. That is, relative time reversal would never be observed between an event and any other event in its absolute past or future. Effect preceeding cause can never actually be observed. If one could, however, send signals faster than light – which one can't of course – one may be able to see cause follow effect which would look as if time were running backward!