

Homework 13
Physics 2130
Modern Physics
Due: Wednesday December 11, 2002

1. T & Z, #13.50.

(a) $M(U)$ = mass of uranium consumed per year. Mass of ^{235}U consumed per year = $M(^{235}\text{U}) = (0.0072)M(U)$ because ^{235}U constitutes only 0.72% of all uranium.

$$\begin{aligned} \text{Energy produced by fission} &= (30\%) \times (\text{number of fissions}) \times (\text{energy per fission}) \\ &= 0.3 \left(\frac{0.0072 M(U) 6.02 \times 10^{23} \text{ particles/mole}}{235 \text{ g/mole}} \right) 200 \text{ MeV/fission} (1.6 \times 10^{-13} \text{ J/MeV}) \\ &= 1.77 \times 10^8 \text{ J/g} \times M(U). \end{aligned}$$

$$\text{Energy per year, power plant} = 10^9 \text{ W} (3.15 \times 10^7 \text{ s/y}) = 3.15 \times 10^{16} \text{ J/y.}$$

$$1.77 \times 10^8 \text{ J/g} \times M(U) = 3.15 \times 10^{16} \text{ J/y}$$

$$M(U) = 2 \times 10^8 \text{ g} = 2 \times 10^5 \text{ kg} = 200 \text{ tonnes (metric tons)}$$

(b) Mass of coal = $M(\text{coal})$.

$$\text{Energy produced by coal} = (0.40)M(\text{coal})(33 \times 10^6 \text{ J/kg})$$

$$= (1.31 \times 10^7 \text{ J/kg})M(\text{coal}) = 3.15 \times 10^{16} \text{ J}$$

$$M(\text{coal}) = 2.4 \times 10^6 \text{ tonnes}$$

(c) $M(\text{coal}) = 24,000$ box car loads ~ 240 trains.

(d) $M(U) = 2$ box car loads < 1 train.

2. T & Z, #13.51.

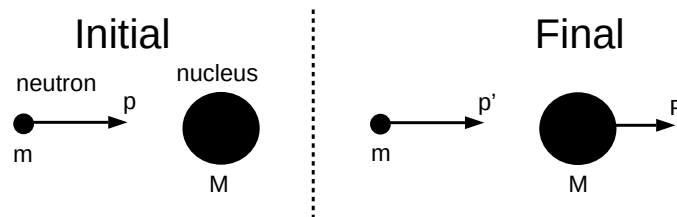


Figure 1:

$$\text{Conservation of momentum} \quad P = p - p' \quad (1)$$

$$\text{Conservation of kinetic energy} \quad \frac{P^2}{2M} = \frac{p^2}{2m} + \frac{p'^2}{2m} \rightarrow \mu P^2 = p^2 - p'^2 \quad (2)$$

Note that kinetic energy is conserved because this is an elastic collision. Dividing equation (2) by (1) and adding the result to equation (1) yields:

$$(1 + \mu)P = p \rightarrow P = \frac{2p}{1 + \mu} \quad (3)$$

The kinetic energy lost by the neutron, ΔK , will be just equal to the kinetic energy of the recoiling nucleus:

$$\begin{aligned}\Delta K &= \frac{P^2}{2M} \stackrel{(3)}{=} \frac{4p^2}{2M(1+\mu)^2} \\ \frac{\Delta K}{K} &= \frac{4p^2}{2M(1+\mu)^2} \frac{2m}{p^2} = \frac{4m}{M} \frac{1}{(1+\mu)^2} = \frac{4\mu}{(1+\mu)^2} = f(\mu)\end{aligned}\quad (4)$$

where we have noted that the relative kinetic energy loss of the incident photon is a function of mu and call that function $f(\mu)$.

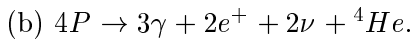
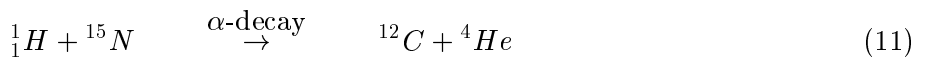
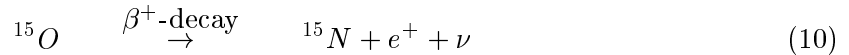
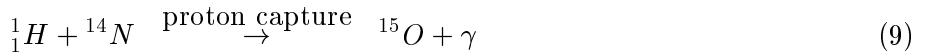
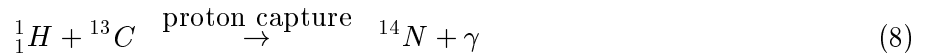
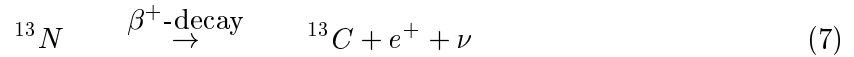
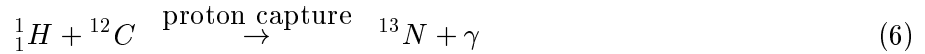
(b) To maximize equation (4) with respect to μ we take the derivative and set it equal to 0:

$$\frac{df(\mu)}{d\mu} = \frac{4(1-\mu)}{(1+\mu)^3} = 0 \rightarrow \mu = 1; \quad (5)$$

that is, energy loss is maximum when the two masses are equal.

(c) Deuterium: $\mu = 1/2 \rightarrow \Delta K/K = 0.89$. Carbon: $\mu = 1/12 \rightarrow \Delta K/K = 0.28$. Note that if $\Delta K/K = 1$, the neutron is brought to a complete halt during the collision.

3. T & Z, #13.55.



(c) In the CNO cycle, the highest Coulombic barrier is in the last step, reaction (11):

$$U_{\text{CNO}} = \frac{7ke^2}{r_1}.$$

In the $p-p$ cycle, the highest barrier is for ${}^3He + {}^3He \rightarrow {}^4He + 2p$:

$$U_{pp} = \frac{4ke^2}{r_2}.$$

If $r_1 = r_2$, $U_{\text{CNO}}/U_{pp} = 7/4$. Since the average kinetic energy is proportional to temperature:

$$\frac{T_{\text{CNO}}}{T_{pp}} \approx \frac{7}{4}.$$

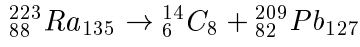
4. T & Z, #13.56.

(3 pts) T & Z #13.56.

$$a = 2\pi ke^2 \sqrt{2m}/\hbar = 2\pi(1.44 \text{ MeV fm}) \sqrt{2(4.003)(931.5)}/197 \text{ MeV fm} = 3.97 \text{ (MeV)}^{1/2}$$

$$b = 8\sqrt{mke^2}/\hbar = 8\sqrt{4.003(931.5 \text{ MeV})(1.44 \text{ MeV fm})}/197 \text{ MeV fm} = 2.98 \text{ (fm)}^{-1/2}$$

5. T & Z, #13.61.



$$(a) K_C = \Delta Mc^2 = [m_{\text{atom}}({}^{223}\text{Ra}) - m_{\text{atom}}({}^{14}\text{C}) - m_{\text{atom}}({}^{209}\text{Pb})] c^2 = 0.0341 \text{uc}^2 (931.5 \text{ MeV}/c^2) = 31.75 \text{ MeV}.$$

(b) In this decay, $U(r) = 6Zke^2/r$ rather than $2Zke^2/r$ because there are 6 positive charges in the nucleus of the carbon atom. Also, $m_{\text{nucl}}({}^{14}\text{C}) = 3.5m_{\text{nucl}}(\alpha)$ because the α particle is ${}^4_2\text{He}_2$.

Equation (13.77) can be used with the following substitutions:

$$\begin{aligned} Z &\rightarrow Z' = 3Z = 3(82) \\ m &\rightarrow m' = 3.5m \\ a &\rightarrow a' = \sqrt{3.5}a \approx \sqrt{3.5}(4) \text{ (MeV)}^{1/2} \\ b &\rightarrow b' = \sqrt{3.5}b \approx \sqrt{3.5}(3) \text{ (fm)}^{-1/2} \end{aligned}$$

where $a \approx 4 \text{ (MeV)}^{1/2}$, $b \approx 3 \text{ (fm)}^{-1/2}$, and we need to use $Z = 82$ for the number of protons in the offspring.

(c) We're told that $R = 7.5 \text{ fm}$ and $K = 31.75 \text{ MeV}$.

$$\begin{aligned} P({}^{14}\text{C}) &= e^{-a'Z'K^{-1/2}+b'(Z'R)^{1/2}} = \exp \left[-\frac{\sqrt{3.5}(4)(3)(82)}{\sqrt{31.75}} + \sqrt{3.5}3(82(7.5))^{1/2} \right] \\ &= 6 \times 10^{-38}. \end{aligned}$$

Similarly, using $a, b, Z, K_\alpha = 6 \text{ MeV}$ rather than a', b', Z', K_C gives:

$$P(\alpha) = 1.6 \times 10^{-28} \rightarrow \frac{P({}^{14}\text{C})}{P(\alpha)} \approx 3 \times 10^{-10}$$

α -decay is about 10 orders of magnitude more probable.

Extra Credit. T & Z #13.60.

We need to do the following integral:

$$I = \int_{x_0}^{x_1} \alpha(x) dx, \tag{12}$$

where

$$P = e^{-2I} \tag{13}$$

and

$$\alpha(x) = \frac{\sqrt{2m(U(x) - E)}}{\hbar} \tag{14}$$

$$U(x) = \frac{2Zke^2}{x} \tag{15}$$

$$E = K = \frac{2Zke^2}{x_1} \tag{16}$$

After a little algebra using equations (14) - (16), we see that:

$$\begin{aligned}\alpha^2(x) &= \frac{4mZke^2}{\hbar^2} \left(\frac{1}{x} - \frac{1}{x_1} \right) \\ I &\stackrel{(12)}{=} \frac{4mZke^2}{\hbar^2} \int_{x_0}^{x_1} \left(\frac{1}{x} - \frac{1}{x_1} \right) dx = \frac{2e}{\hbar} \sqrt{\frac{mZk}{x_1}} \int_{x_0}^{x_1} \sqrt{\frac{x_1}{x} - 1} dx \\ &= \frac{2e}{\hbar} \sqrt{\frac{mZk}{x_1}} J\end{aligned}\tag{17}$$

$$J = \int_{x_0}^{x_1} \sqrt{\frac{x_1}{x} - 1} dx\tag{18}$$

So, doing integral I reduces to doing integral J , in which we use the following change of variables:

$$\begin{aligned}x &= x_1 \cos^2 \theta \quad \text{or} \quad \theta = \cos^{-1} \left(\frac{x}{x_1} \right)^{1/2} \\ dx &= -x_1 (2 \cos \theta \sin \theta) d\theta \\ x &= x_1 \rightarrow \theta = 0 \\ x &= x_0 \rightarrow \theta = \theta_0.\end{aligned}\tag{19}$$

$$\begin{aligned}J &= \int_{x_0}^{x_1} \sqrt{\frac{x_1}{x} - 1} dx \\ &= - \int_{\theta_0}^0 \sqrt{\frac{1}{\cos^2 \theta} - 1} x_1 (2 \cos \theta \sin \theta) d\theta \\ &= \int_0^{\theta_0} \sqrt{\frac{1 - \cos^2 \theta}{\cos^2 \theta}} (2x_1 \cos \theta \sin \theta) d\theta \\ &= 2x_1 \int_0^{\theta_0} \sqrt{\frac{\sin^2 \theta}{\cos^2 \theta}} (\cos \theta \sin \theta) d\theta \\ &= 2x_1 \int_0^{\theta_0} \sin^2 \theta d\theta = x_1 \int_0^{\theta_0} (1 - \cos 2\theta) d\theta \\ &= x_1 \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\theta_0} = x_1 \left(\theta_0 - \frac{1}{2} \sin 2\theta_0 \right) \\ &= x_1 \left(\theta_0 - \frac{1}{2} 2 \sin \theta_0 \cos \theta_0 \right)\end{aligned}\tag{20}$$

Now, by equation (19, $\theta_0 = \cos^{-1} (x_0/x_1)$), which for $x_0 \ll x_1$ is very close to $\pi/2$ so $\theta_0 \approx \pi/2 - \sqrt{x_0/x_1}$ and $\sin \theta_0 \approx 1$ and $\cos \theta_0 \approx \sqrt{x_0/x_1}$. With these values, we can rewrite equation (20) as follows:

$$\begin{aligned}J &= x_1 \left(\frac{\pi}{2} - 2\sqrt{\frac{x_0}{x_1}} \right) \rightarrow I \stackrel{(17)}{=} \frac{2e}{\hbar} \sqrt{\frac{mZk}{x_1}} x_1 \left(\frac{\pi}{2} - 2\sqrt{\frac{x_0}{x_1}} \right) \\ &= \frac{e\pi}{\hbar} \sqrt{mZkx_1} - \frac{4e}{\hbar} \sqrt{mZkx_0} \\ &= \frac{\pi ke^2 \sqrt{2m}}{\hbar} ZK^{-1/2} - \frac{4}{\hbar} \sqrt{mke^2} (ZR)^{1/2}\end{aligned}\tag{21}$$

where in the final equality we let $x_0 = R$ and $x_1 = 2Zke^2/K$.

Recalling that $P = \exp(-2I)$, equation (21) can be seen to give us our final result:

$$P = \exp \left[-\frac{2\pi k e^2 \sqrt{2m}}{\hbar} Z K^{-1/2} + \frac{8}{\hbar} \sqrt{m k e^2} (Z R)^{1/2} \right]$$