

Homework 10
Physics 2130
Modern Physics
Due: Wednesday November 20, 2002

1. **Some math.**

(a) Taylor and Zafiratos, problem 8.11.

$$\begin{aligned}a \cos \omega t + b \sin \omega t &= A \sin(\omega t + \phi) \\&= A (\sin \omega t \cos \phi + \cos \omega t \sin \phi) \\&= (A \cos \phi) \sin \omega t + (A \sin \phi) \cos \omega t\end{aligned}$$

So, $a = A \sin \phi$ and $b = A \cos \phi$.

Let $t' = t + \alpha \Rightarrow t = t' - \alpha$.

$$\sin(\omega t + \phi) = \sin[\omega(t' - \alpha) + \phi] = \sin[\omega t' + (\phi - \alpha/\omega)] = \sin \omega t',$$

only if $\alpha = \phi/\omega$.

Therefore, $t' = t + \phi/\omega$.

(b) Taylor and Zafiratos, problem 8.12.

$$\begin{aligned}e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \\&= 1 + i\theta - \theta^2/2! - i\theta^3/3! + \theta^4/4! + \dots \\&= (1 - \theta^2/2! + \theta^4/4! + \dots) + i(\theta - \theta^3/3! + \dots) \\&= \cos \theta + i \sin \theta\end{aligned}$$

(c). Taylor and Zafiratos, problem 8.22.

(a) Test to see if $\psi'' = -k^2\psi$ for several trial forms of $\psi(x)$.

$$\begin{aligned}\psi(x) = \sin kx &\Rightarrow \frac{d^2}{dx^2} \sin kx = \frac{d}{dx} (-k \cos kx) = -k^2 \sin kx = -k^2\psi \\ \psi(x) = \cos kx &\Rightarrow \frac{d^2}{dx^2} \cos kx = \frac{d}{dx} (-k \sin kx) = -k^2 \cos kx = -k^2\psi \\ \psi(x) = e^{ikx} &\Rightarrow \frac{d^2}{dx^2} e^{ikx} = \frac{d}{dx} (ike^{ikx}) = -k^2 e^{ikx} = -k^2\psi\end{aligned}$$

(b) $e^{ikx} = \cos x + i \sin kx$.

2. Taylor and Zafiratos, problem 8.16.

Note $a = 1 \text{ cm} = 10^7 \text{ nm}$.

$$\begin{aligned} E_n &= n^2 \frac{\pi^2 \hbar^2}{2ma^2} = n^2 \frac{\pi^2 (\hbar c)^2}{2(mc^2)a^2} = n^2 \frac{(3.1416)^2 197^2 \text{ eV}^2 \text{ nm}^2}{2(.511 \times 10^6 \text{ eV } 10^{14} \text{ nm}^2)} \\ &= n^2 3.75 \times 10^{-15} \text{ eV} \end{aligned}$$

$E_2 - E_1 = (2^2 - 1) 3.75 \times 10^{-15} \text{ eV} = 1.12 \times 10^{-14} \text{ eV}$. This is very small because a is large, so energy levels will behave almost classically.

3. Taylor and Zafiratos, problem 8.31. We're asked questions about the ground-state wavefunction, so $n = 1$ and we let $\psi(x) = \psi_1(x)$ for simplicity.

$$\begin{aligned} P(b, c) &= \int_b^c |\psi(x)|^2 dx \\ \psi(x) &= \sqrt{\frac{2}{a}} \sin kx = \sqrt{\frac{2}{a}} \sin(\pi x/a) \\ |\psi(x)|^2 &= \frac{2}{a} \sin^2(\pi x/a) \end{aligned}$$

Before doing specific integrals, let's do the integral in general for endpoints given by (b, c) and then we can substitute particular values for b and c in the subparts of the problem. To do this integral we use the trig identity that $\sin^2 \alpha = (1 - \cos 2\alpha)/2$ and recall that the integral of $\cos \alpha x$ is: $\int \cos \alpha x dx = (\sin \alpha x)/\alpha$.

$$\begin{aligned} P(b, c) &= \frac{2}{a} \int_0^c \sin^2(\pi x/a) = \frac{1}{a} \int_c^c [1 - \cos(2\pi x/a)] \\ &= \frac{1}{a} x \Big|_b^c - \frac{1}{a} \left[\frac{a}{2\pi} \sin(2\pi x/a) \right] \Big|_b^c \\ &= \frac{c - b}{a} - \frac{1}{2\pi} [\sin(2\pi c/a) - \sin(2\pi b/a)] \end{aligned}$$

(a)

$$P(0, c) = \frac{2}{a} \int_0^c \sin^2(\pi x/a) = \frac{c}{a} - \frac{1}{2\pi} \sin(2\pi c/a)$$

(b)

$$P(0, a) = \frac{2}{a} \int_0^a \sin^2(\pi x/a) = \frac{a}{a} - \frac{1}{2\pi} \sin(2\pi a/a) = 1 - \frac{1}{2\pi} \sin 2\pi = 1$$

(c)

$$P(0, a/2) = \frac{2}{a} \int_0^{a/2} \sin^2(\pi x/a) = \frac{a}{2a} = \frac{1}{2}$$

(d)

$$P(0, a/4) = \frac{2}{a} \int_0^{a/4} \sin^2(\pi x/a) = \frac{a}{4a} - \frac{1}{2\pi} \sin(\pi/2) = \frac{1}{4} - \frac{1}{2\pi} \approx 0.091$$

(d) $\sin^2(\pi x/a)$ is not linear.

4. Taylor and Zafra, problem 8.33.

Classical turning points occur where the energy and potential are equal:

$$E = U_0 \left(1 - e^{x^2/a^2}\right) \rightarrow x = a \left[\ln \left| 1 - \frac{E}{U_0} \right| \right]^{1/2} = a \left[\ln \left(\frac{U_0}{U_0 - E} \right) \right]^{1/2}$$

5. **Traveling waves.**

(a) Consider a traveling wave on a string as follows moving in the $+x$ -direction: $y(x, t) = \sin(kx - \omega t)$. Show that $y(x, t)$ is a solution of the 1-D wave equation for a string; where the 1-D wave equation is:

$$\begin{aligned} \frac{\partial^2 y(x, t)}{\partial x^2} &= \frac{1}{c^2} \frac{\partial^2 y(x, t)}{\partial t^2} \\ \frac{\partial^2 y}{\partial x^2} &= -k^2 \sin(kx - \omega t) = -k^2 y \\ \frac{\partial^2 y}{\partial t^2} &= -\omega^2 \sin(kx - \omega t) = -\omega^2 y \end{aligned} \tag{1}$$

from which the desired result follows immediately because $c = \omega/k$.

(b) Now consider a wavefunction for a free particle in which $U = 0$ also traveling in the $+x$ -direction:

$$\Psi(x, t) = e^{i(\sqrt{2mE}x - Et)/\hbar}$$

Show that $\Psi(x, t)$ is a solution of the time-dependent Schrodinger equation with zero potential:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} = i\hbar \frac{\partial \Psi(x, t)}{\partial t}$$

Note that there were two typos in this problem as originally stated. One correct answer would be to say that, as stated, the functional form

of $\Psi(x, t)$ given is not a solution to the differential equation given. A better answer would be to have corrected the typos..... In either case, you'll get full credit.

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial x^2} &= \frac{-2mE}{\hbar} \Psi \\ \frac{\partial \Psi}{\partial t} &= \frac{-iE}{\hbar} \Psi\end{aligned}\tag{2}$$

from which the result follows immediately.

6. **Schrodinger equation with a non-zero potential.** Given that the quantum mechanical wavefunction $\Psi(x, t) = \exp[i(kx - \omega t)]$ and the potential energy $U = \text{constant} \neq 0$, following the procedure in the notes to show that the time-dependent Schrodinger equation for a particle subjected to this potential is:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

Assume that the particle is subjected to a constant potential energy $U(x) = U_0$. If we have a traveling wave, then

$$\Psi(x, t) = e^{i(kx - \omega t)}\tag{3}$$

Irrespective of the potential $\omega = E/\hbar$, but the wavenumber depends on the potential as follows:

$$K = E - U_0 = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \Rightarrow k^2 = \frac{2mK}{\hbar^2} \Rightarrow k = \frac{\sqrt{2m(E - U_0)}}{\hbar}$$

which we previously called β in the notes. Equation (3) then becomes

$$\Psi(x, t) = e^{i(\sqrt{2m(E-U_0)}x - Et)/\hbar}$$

Consider the space and time derivatives of this form of the wavefunction:

$$\frac{\partial \Psi(x, t)}{\partial t} = -\frac{iE}{\hbar} e^{i(\sqrt{2m(E-U_0)}x - Et)/\hbar} = -\frac{iE}{\hbar} \Psi(x, t)\tag{4}$$

$$\begin{aligned}\frac{\partial \Psi(x, t)}{\partial x} &= \frac{i\sqrt{2m(E - U_0)}}{\hbar} e^{i(\sqrt{2m(E-U_0)}x - Et)/\hbar} = \frac{i\sqrt{2m(E - U_0)}}{\hbar} \Psi(x, t) \\ \frac{\partial^2 \Psi(x, t)}{\partial x^2} &= -\frac{2m(E - U_0)}{\hbar^2} \Psi(x, t)\end{aligned}\tag{5}$$

Inspection of equations (4) and (5) reveals that:

$$\begin{aligned}i\hbar \frac{\partial \Psi(x, t)}{\partial t} &\stackrel{(4)}{=} E\Psi(x, t) \\ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + U_0 \Psi(x, t) &\stackrel{(5)}{=} E\Psi(x, t)\end{aligned}$$

Combining these results we get the Schrodinger equation for a particle subjected to a constant potential U_0 :

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + U_0 \Psi(x, t) = i\hbar \frac{\partial \Psi(x, t)}{\partial t}$$

7. **Separation of variables applied to the Schrodinger equation.** Given the time-dependent Schrodinger equation for the wavefunction $\Psi(x, t)$:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U(x) \Psi(x, t) = i\hbar \frac{\partial \Psi}{\partial t}. \quad (6)$$

Let

$$\Psi(x, t) = \psi(x)T(t). \quad (7)$$

Using separation of variables, derive the time-independent Schrodinger equation for the spatial wavefunction $\psi(x)$:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + U(x) \psi(x) = E \psi(x) \quad (8)$$

Remember to let E be the separation constant.

To substitute equation (7) into (6) we'll need the following derivatives:

$$\begin{aligned} \frac{\partial^2 \Psi(x, t)}{\partial x^2} &= T(t) \frac{\partial^2 \psi(x)}{\partial x^2} = T(t) \frac{d^2 \psi(x)}{dx^2} \\ \frac{\partial \Psi(x, t)}{\partial t} &= \psi(x) \frac{\partial T(t)}{\partial t} = \psi(x) \frac{dT(t)}{dt} \end{aligned}$$

The partial derivatives are replaced with ordinary derivatives at the end of each equation because the function that's being differentiated is a function only of a single variable. So, we substitute these equations into equation (6):

$$-\frac{\hbar^2}{2m} T(t) \frac{d^2 \psi(x)}{dx^2} + U(x) \psi(x) T(t) = i\hbar \psi(x) \frac{dT(t)}{dt}$$

Next we rearrange to separate the functions of t and x by dividing both sides of the equation by $\psi(x)T(t)$:

$$-\frac{\hbar^2}{2m} \frac{1}{\psi(x)} \frac{d^2 \psi(x)}{dx^2} + U(x) = i\hbar \frac{1}{T(t)} \frac{dT(t)}{dt} \quad (9)$$

Note that in equation (9), we have a function of x equal to a function of t . The only way this can be true in general is if both sides of the equation are equal to the same constant. Because both sides of the equation have the units

of energy, let's let the constant that both sides of equation (9) equals be called E . If we do that, equation (6) divides into two equations:

$$-\frac{\hbar^2}{2m} \frac{1}{\psi(x)} \frac{d^2\psi(x)}{dx^2} + U(x) = E$$

$$i\hbar \frac{1}{T(t)} \frac{dT(t)}{dt} = E$$

Multiplying the first equation by $\psi(x)$, we get the desired result, equation (8).

8. **Sum of wavefunctions.** By the Principle of Superposition, any superposition (or sum) of wavefunctions is a possible wavefunction. Suppose that a particle in a box is in a superposition of the ground state and the first excited state, with an amplitude of $1/2$ for the former and $\sqrt{3}/2$ for the latter. The wavefunction for this superposition is:

$$\begin{aligned}\Psi(x, t) &= \frac{1}{2}\Psi_1(x, t) + \frac{\sqrt{3}}{2}\Psi_2(x, t) \\ &= \frac{1}{2}\sqrt{\frac{2}{a}}e^{-iE_1t/\hbar} \sin\left(\frac{\pi x}{a}\right) + \frac{\sqrt{3}}{2}\sqrt{\frac{2}{a}}e^{-iE_2t/\hbar} \sin\left(\frac{2\pi x}{a}\right)\end{aligned}$$

Find the probability distribution for this wavefunction; i.e., find $|\Psi(x, t)|^2$.

This is an exercise in multiplying complex numbers. Let's do this in the abstract first, and then substitute the actual forms given in this problem. Abstractly, we're to find the modulus-squared of the sum of two complex numbers. So, let $\Psi = a\Psi_1 + b\Psi_2 = ar_1 \exp(i\phi_1) + br_2 \exp(i\phi_2)$. Therefore,

$$\begin{aligned}|\Psi|^2 = \Psi\Psi^* &= ar_1e^{i\phi_1} + br_2e^{i\phi_2} \\ &\times ar_1e^{-i\phi_1} + br_2e^{-i\phi_2} \\ &= \overline{a^2r_1^2 + b^2r_2^2 + (abr_1r_2e^{i(\phi_1-\phi_2)} + abr_1r_2e^{-i(\phi_1-\phi_2)})} \\ &= a^2r_1^2 + b^2r_2^2 + 2abr_1r_2 \cos(\phi_1 - \phi_2)\end{aligned}\tag{10}$$

In this problem we have the following substitutions:

$$\begin{aligned}a &= 1/2 \\ b &= \sqrt{3}/2 \\ r_1 &= \sqrt{2/a} \sin(\pi x/a) \\ r_2 &= \sqrt{2/a} \sin(2\pi x/a) \\ \phi_1 &= -E_1/\hbar \\ \phi_2 &= -E_2/\hbar\end{aligned}$$

So, substituting into equation (10):

$$|\Psi|^2 = \frac{1}{2a} \sin^2(\pi x/a) + \frac{3}{2a} \sin^2(2\pi x/a) + \frac{\sqrt{3}}{a} \sin(\pi x/a) \sin(2\pi x/a) \cos(E_2 - E_1) t/\hbar$$